A DYNAMIC BIRTH-DEATH MODEL VIA INTRINSIC LINKAGE

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Abstract

An analytically useful dynamic birth-death model is advanced based on Intrinsic Linkage, i.e. the linear relationship between Leslie matrix and population functions shown in Eq(1). The key parameter of the model, w, captures the essential contribution of the subordinate components of a population projection matrix, and enables population composition over time to be found from the sequence of past Leslie matrix intrinsic growth rates. When those intrinsic growth rates follow a polynomial, exponential, or cyclical pattern over time, model population composition can be expressed analytically in closed form. Eq(30) provides a simple sufficiency condition for the existence of demographically valid projection matrices. Three numerical illustrations show model values and relationships under metastable, cyclically stationary, and cyclically stable patterns of change. The Intrinsic Linkage approach extends current techniques for dynamic modeling, revealing new relationships between population structures and the changing vital rates that generate them.

KEY WORDS: birth-death models, Leslie matrices, dynamic populations, cyclical stability

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The dominant mathematical model in demography, the stable population, is based on constant age-specific rates of birth and death. In actual populations, age-specific vital rates regularly change over time, often substantially. The ability to model such changes, and to analytically specify their implications for how the size and age structure of a population change over time, remains a challenge for mathematical demography.

Previous work has begun to explore dynamic models, that is models with time-varying vital rates. Coale (1972) and Lee (1974) were pioneering works on birth-death models. Bongaarts and Feeney (2002) advanced a flexible dynamic mortality-only model. Schoen and Kim (1994 PAA) and Schoen and Jonsson (2003) developed the metastable (or quadratic hyperstable) model that generalized the stable model, allowing fertility to change exponentially over both age and time. Schoen (2006, Chapter 7) discussed several other approaches to modeling changing rates, including Intrinsically Dynamic and "hyperstable" models. Yet despite the progress to date, there is no generally applicable analytical solution for the age structure produced by an arbitrarily changing set of vital rates.

The present paper advances a new approach to dynamic modeling that is applicable to a broad range of vital rate trajectories. This “Intrinsic Linkage” approach is based on the intuitively plausible idea that the age composition of the time t population projection (or Leslie) matrix can be represented as a weighted average of the model population age compositions at times t and t−1. The Intrinsic Linkage assumption leads to new, analytically tractable relationships between the trajectory of Leslie matrix intrinsic growth rates and the size and age composition of the model at any point in time.

THE BASIC INTRINSIC LINKAGE EQUATIONS

Consider an n-age group discrete birth-death population model, and let $\mathbf{A_t}$ be the population projection (Leslie) matrix that takes the model population from time t−1 to time t. Denote the intrinsic or dominant root (or dominant eigenvalue) of $\mathbf{A_t}$ by λ_t, and let λ_{Pt} represent the growth rate of model population births (i.e. members of the first age group) from time t−1 to time t. Assuming no mortality before the end of childbearing, the intrinsic age composition implied by $\mathbf{A_t}$ follows from λ_t. With one person in the first age group, there are λ_t^{-1} persons in the second age group, λ_t^{-2} persons in the third, and so on.

It is reasonable to assume that λ_t, the long term growth rate of births implied by $\mathbf{A_t}$, modifies $\lambda_{P,t-1}$ to produce λ_{Pt}. Letting that relationship be linear, there must be some scalar value, w_t, such that

$$\lambda_{Pt} = \lambda_t(1- w_t) + \lambda_{P,t-1}\, w_t \tag{1}$$

Eq(1) must hold, as any number can be written as a linear combination of any 2 given numbers. Formally, one can always find w_t from the model and Leslie values (i.e. from $\lambda_{P,t-1}$, λ_t, and λ_{Pt}), hence Eq(1) applies to any population and w_t can be seen as a parameter that describes how λ_t and $\lambda_{P,t-1}$ yield λ_{Pt}.

Eq(1) can be readily cumulated over time to express λ_{Pt} in terms of the sequence of λ_t and w_t and the "time 0" model value λ_{P0}. Straightforward algebra yields

$$\lambda_{Pt} = [\ \lambda_{P0} \prod_{j=1}^{t} w_j\ ] + \sum_{j=1}^{t} \lambda_j\,(1-w_j)\ [\ \prod_{i=j+1}^{t} w_i\ ] \tag{2}$$

In a demographically realistic model, λ_{Pt} must be finite and positive, so the product of the w_t must be bounded at all times. With the population projection matrices being, as usual, non-negative and primitive, weak ergodicity applies (Schoen 2006, Chap. 2) and the initial growth and composition are eventually forgotten. Thus the first term on the right in Eq(2) must disappear as t becomes large, and the product of the w_j must become increasingly smaller, so that the sum in Eq(2) converges. Eq(2) is a generally valid expression that relates the time t growth rate of births in any model population to the w_t weights and the past sequence of Leslie matrix growth rates.

If parameter w is *constant over time* and t is large, Eq(2) is changed to a form that is no longer simply representational, and the Intrinsic Linkage relationship becomes

$$\lambda_{Pt} = (1-w)\sum_{j=1}^{t} \lambda_j\, w^{t-j} = (1-w)\,[\lambda_t + w\,\lambda_{t-1} + w^2\,\lambda_{t-2} + w^3\,\lambda_{t-3} + \ldots + w^{t-1}\,\lambda_1] \qquad (3)$$

Eq (3) shows that at large t, the growth in the number of model births between times t−1 and t (or the relative size of the number in the first age group to that in the second at time t) is the sum of a convergent power series in w, where values of $|w|<1$ to increasingly higher powers are applied to earlier period Leslie matrix growth rates. Since

$$1 - w = 1 / [\, 1 + w + w^2 + w^3 + \ldots] \qquad (4)$$

Eq(3) shows that λ_{Pt} is a weighted average of λ_j values, where the most recent λ_j have the greatest weight.

Equation (3) is a new result, and is significant as it provides an intuitively meaningful relationship between model and Leslie composition/growth rates. If the λ_j are constant, then $\lambda = \lambda_{Pt}$ and the model is stable. In general, the power series in Eq(3) describes how earlier λ_j get increasingly smaller weights, and thus become "forgotten" over time. If there is a regularity in the λ_j that allows an algebraic summation, then Eq(3) can provide a closed form relationship between the λ_j and λ_{Pj} at all time points.

RELATIONSHIPS UNDER INTRINSIC LINKAGE

Let us consider some relationships between patterned sequences of Leslie matrix roots and model population growth. Specifically, we consider cases where w is constant over time and the λ_j sequence is linear, quadratic, and metastable.

The case of Leslie matrices with linearly changing roots and constant w.

It is useful to rewrite Eq(3) as a difference in λ_t values using Eqs (3) and (4). That yields

$$\lambda_{Pt} = \lambda_t + \sum_{j=1}^{t-1} (\lambda_{t-j} - \lambda_{t-j+1}) \, w^t \qquad (5)$$

Now let the λ_j sequence change linearly, i.e. let

$$\lambda_t = b_0 + b_1 t \qquad (6)$$

for constants b_0 and b_1. Then Eq(3) becomes

$$\lambda_{Pt} = \lambda_t - b_1 w / (1-w) \qquad (7)$$

Thus, when w is constant, λ_{Pt} changes linearly with λ_t, with the 2 growth rates separated by a constant amount at every time point.

To see how the growth rate of model population births, λ_{Pt}, varies with parameter w in the linear case, we can take the partial derivative of Eq(7) and write

$$\partial \lambda_{Pt} / \partial w = - b_1 / (1-w)^2 \qquad (8)$$

If $b_1 > 0$, i.e. when λ_{Pt} is increasing, then an increase in w implies a reduction in λ_{Pt}.

The case of Leslie matrices with quadratically changing roots.

Let λ_t change quadratically, i.e. let

$$\lambda_t = b_0 + b_1 t + b_2 t^2 \qquad (9)$$

Then λ_{Pt} can be found using Eqs (3) - (5) and the Maple derived summation relationship

$$S_2 = \sum_{j=1}^{t} b_2\, j\, w^{t-j} = [\, b_2 t\,(1-w) - w b_2\,] / (1-w)^2 \qquad (10)$$

The result is

$$\lambda_{Pt} = \lambda_t - w b_1/(1-w) - w b_2\, [\, 2t(1-w)-(1+w)\,] / (1-w)^2 \qquad (11)$$

Here, the difference between λ_t and λ_{Pt} changes linearly with time.

The λ_{Pt} associated with cubic and higher powers of change in λ_t can be found using the above approach, but at the price of more complex relationships. In principle, at least, Eq(3) can yield an explicit relationship for λ_{Pt} for any polynomial λ_t.

The case of Leslie matrices from a metastable population model.

The difference formulation of Eq(5) leads to an explicit solution for λ_{Pt} in the metastable model, that is the case where

$$\lambda_t = c\, k^t \qquad (12)$$

and c and k are constants. When k=1, the population is stable. It follows from Eq(3) that

$$\lambda_{Pt} = c\, k^t\, [\, k\,(1-w) / (k-w)\,] \qquad (13)$$

or that λ_{Pt} is ck^t times a constant factor [previously denoted λ_s by Schoen (2006:134)] that depends on w and on exponential growth parameter k. The constant w Intrinsic Linkage formulation thus leads to a new, closed form expression for λ_s, specifically

$$\lambda_s = k\,(1-w) / (k-w) \qquad (14)$$

To consider how the metastable λ_{Pt} varies with parameter w, we differentiate Eq(13) with respect to w and find

$$\partial\, \lambda_{Pt} / \partial w = -c\, k^{t+1}\, (k-1) / (k-w)^2 \qquad (15)$$

If k>1 so that λ_{Pt} is increasing over time, then an increase in w decreases λ_{Pt}. As in the linear case, a change in w moves λ_{Pt} in the opposite direction.

CYCLICALLY STABLE POPULATION MODELS WITH INTRINSIC LINKAGE

Cyclically stable populations arise naturally in a number of applications. Their analysis was pioneered by Skellam (1967), with significant work by Namboodiri (1969) and Tuljapurkar (1985; 1990). The Intrinsic Linkage approach is well suited to analyzing cyclical populations, and equations (2) and (3) yield new and explicit solutions for the λ_{Pt} trajectory when λ_t varies cyclically.

The case where cycle length is 2.

Consider the simplest case under Eq(2), where λ_t alternates in value between λ_1 (when t is odd) and λ_2 (when t is even). Then, with w_1 associated with λ_1 and w_2 associated with λ_2, Eq(1) implies

$$\lambda_{P1} = \lambda_1 (1-w_1) + \lambda_{P2} w_1$$

$$\lambda_{P2} = \lambda_2(1-w_2) + \lambda_{P1} w_2 \tag{16}$$

Using Eq(2) with t large, and summing the terms in λ_1 and λ_2 separately, gives the solutions

$$\lambda_{P1} = [\ \lambda_1(1-w_1) + \lambda_2 w_1 (1-w_2)\ ] / [\ 1 - w_1w_2\ ]$$

$$\lambda_{P2} = [\ \lambda_2(1-w_2) + \lambda_1 w_2 (1-w_1)\ ] / [\ 1 - w_1w_2\ ] \tag{17}$$

Under Eq(3), with w constant, the solutions in Eq(17) reduce to

$$\lambda_{P1} = [\ \lambda_1 + \lambda_2 w\ ] / [\ 1 + w\ ]$$

$$\lambda_{P2} = [\ \lambda_2 + \lambda_1 w\ ] / [\ 1 + w\ ] \tag{18}$$

with $\lambda_{P1} + \lambda_{P2} = \lambda_1 + \lambda_2$.

The case where cycle length is 3 and w is constant.

When each cycle spans 3 intervals, the 3 constant w specifying equations are

$$\lambda_{P1} = \lambda_1 (1-w) + \lambda_{P3} w$$

$$\lambda_{P2} = \lambda_2(1-w) + \lambda_{P1} w$$

$$\lambda_{P3} = \lambda_3(1-w) + \lambda_{P2}\, w \tag{19}$$

The solutions for the growth rates of model births are then

$$\lambda_{P1} = (\lambda_1 + w\,\lambda_3 + w^2\,\lambda_2) / (1 + w + w^2)$$

$$\lambda_{P2} = (\lambda_2 + w\,\lambda_1 + w^2\,\lambda_3) / (1 + w + w^2)$$

$$\lambda_{P3} = (\lambda_3 + w\,\lambda_2 + w^2\,\lambda_1) / (1 + w + w^2) \tag{20}$$

which implies $\lambda_{P1} + \lambda_{P2} + \lambda_{P3} = \lambda_1 + \lambda_2 + \lambda_3$.

The general case where cycle length is m and w is constant.

The above approach readily generalizes to cycle lengths of m intervals, where m is any positive integer. With w constant, the m specifying equations are

$$\lambda_{P1} = \lambda_1\,(1-w) + \lambda_{Pm}\, w$$

$$\lambda_{P2} = \lambda_2(1-w) + \lambda_{P1}\, w$$

$$\vdots$$

$$\lambda_{Pm} = \lambda_m(1-w) + \lambda_{P,m-1}\, w \tag{21}$$

Note in the first equation that λ_{Pm} precedes λ_{P1}. The straightforward solution can be written

$$\lambda_{Pj} = \sum_{i=1}^{m} w^{i-1}\,\lambda_{j-i+1} \Big/ \Big(\sum_{i=1}^{m} w^{i-1} \Big) \tag{22}$$

where the value of λ_0 is taken to be λ_m [as in Eqs(16)-(18)]. As before, $\sum \lambda_{Pj} = \sum \lambda_j$, with the sums over j ranging from l to m.

Eq(22) shows the new Intrinsic Linkage solutions for population values in cyclically stable models. Under constant w, closed form solutions that are essentially truncated versions of Eq(3) link the growth in the number of model births to the intrinsic growth rates of each cycle's Leslie matrices.

SPECIFYING THE POPULATION PROJECTION MATRICES

The Intrinsic Linkage model is not complete until the underlying sequence of Leslie matrices is fully specified. The best way to approach the specification of $\mathbf{A_t}$, the Leslie matrix that moves the time t−1 model population to time t, is to begin with models that have only 2 reproductive age groups.

Specifying the 2-age group Intrinsic Linkage Leslie matrix.

Let the 2 x 2 Intrinsic Linkage Leslie matrix be written in the form

$$\mathbf{A_t} = \begin{bmatrix} \lambda_t(1-a_t) & \lambda_t^2 a_t \\ 1 & 0 \end{bmatrix} \tag{23}$$

where again λ_t is the dominant eigenvalue of $\mathbf{A_t}$ and a_t is the contribution to the number of model births from the second age group when the dominant eigenvalue is 1. Schoen (2006: 138-39) shows that such a representation is always possible and describes how it can be implemented. With any number of age groups, the coefficients of the λ's (here a_t and $(1-a_t)$) always sum to 1. Strictly speaking, the model provides a "birth" trajectory, as all persons in age group 1 at time t−1 survive to be in age group 2 at time t. Since $\lambda_t > 0$ and all of the elements of the Leslie matrix are non-negative, we must have

$$0 \le a_t \le 1 \tag{24}$$

for a demographically valid population projection matrix.

Let us scale the model so that the population at time t−1 is described by the vector

$$\mathbf{x_{t-1}} = \begin{bmatrix} 1 \\ 1/\lambda_{P,t-1} \end{bmatrix} \tag{25}$$

and the time t model population is given by

$$\mathbf{x_t} = \lambda_{Pt} \begin{bmatrix} 1 \\ 1/\lambda_{Pt} \end{bmatrix} \tag{26}$$

Eqs(23) - (26) must (and do) satisfy the matrix projection relationship

$$\mathbf{x_t} = \mathbf{A_t}\, \mathbf{x_{t-1}} \tag{27}$$

Using those equations and the Intrinsic Linkage relationship of Eq(1), we can find a scalar equation from the first row of matrix Eq(27) and write

$$\lambda_{Pt} = \lambda_t (1 - a_t) + a_t \lambda_t^2 / \lambda_{P,t-1} = \lambda_t (1 - w_t) + \lambda_{P,t-1}\, w_t \tag{28}$$

Eliminating λ_{Pt} and using the last equality we find

$$a_t = (-w_t)\, \lambda_{P,t-1} / \lambda_t \tag{29}$$

Eq(29) provides the 2-age group Intrinsic Linkage solution for a_t, and thus all of the elements of $\mathbf{A_t}$ are determined.

Because a_t must be nonnegative and $\lambda_{P,t-1}$ and λ_t are always positive, Eq(29) indicates that a demographically valid model for time t requires $w_t < 0$. If w_t is 0, then a_t is 0 and $\mathbf{A_t}$ is not primitive. Since the maximum permissible value of a_t is 1, Eq(29) implies the inequality

$$\lambda_t > (-w_t)\, \lambda_{P,t-1} \tag{30}$$

for a valid Leslie matrix. Combining Eqs(29) and (30), a valid time t model must have

$$0 > w_t > -\lambda_t / \lambda_{P,t-1} \tag{31}$$

For a valid 2-age model, the choice of either w_t or λ_t is constrained by Eq(31). Still, at some specific time t, it is possible for the value of w_t to be less than -1.

Models with w constant over time. If w=0, then $\lambda_t = \lambda_{Pt}$ and $\mathbf{A_t}$ is not primitive. Moreover, in order to have convergence in Eq(3), we must have $-1 < w < 0$. Using Eqs(3) and (4) with $w \neq 0$, we can write the inequality

$$\lambda_t > (-w)\,(1-w)\,[\lambda_{t-1} + w\,\lambda_{t-2} + w^2\,\lambda_{t-3} + w^3\,\lambda_{t-4} + ...] \qquad (32)$$

Eq(32) provides a necessary and sufficient condition for a valid constant w, 2-age group Leslie matrix.

In a constant w Intrinsic Linkage model with $-1 < w < 0$, Eq(1) implies that λ_t is always between λ_{Pt} and $\lambda_{P,t-1}$. If $\lambda_t > \lambda_{P,t-1}$, then $\lambda_{Pt} > \lambda_t$, while if $\lambda_t < \lambda_{P,t-1}$, then $\lambda_{Pt} < \lambda_t$. In contrast, if $1>w>0$, then λ_{Pt} is always between λ_t and $\lambda_{P,t-1}$.

With constant w and a known λ_t sequence, closed form solutions for a_t in terms of the λ_t may be possible. For example, consider the 2-age group metastable model of Eqs(12)-(14). Eq(29) leads to

$$a = (-w)\,(1-w) / (k - w) \qquad (33)$$

with parameter a also constant over time. Since in most cases the metastable parameter k is close to 1, $a \approx -w$ and the demographic validity of the model can quickly be established.

Specifying the 3-age group Intrinsic Linkage Leslie matrix.

Leslie matrices with 3 reproductive age groups can provide a reasonable representation of most human population dynamics. Let the first row of 3-age group Leslie matrix $\mathbf{A_t}$ be $[(1-a_t-b_t)\,\lambda_t,\ a_t\,\lambda_t^2,\ b_t\,\lambda_t^3]$, with, as usual, the matrix having ones on the subdiagonal and zeros elsewhere. Again we must have, $0 \le a_t, b_t \le 1$ and $(1-a_t-b_t) \ge 0$. The time t−1 model population can be scaled so that

$$\mathbf{x_{t-1}}' = [\ 1,\ 1/\,\lambda_{P,t-1}\ ,\ 1/(\,\lambda_{P,t-1}\,\lambda_{P,t-2}\,)\ ] \qquad (34)$$

where the prime (') indicates the transposition from a column vector to a row vector.

The first row of projection Eq(27) then yields the equation

$$(-w_t) / \lambda_t = a_t / \lambda_{P,t-1} + b_t\,[\,(\lambda_t^2 - \lambda_{P,t-1}\,\lambda_{P,t-2}\,) / (\lambda_{P,t-1}\,\lambda_{P,t-2}\{\,\lambda_t - \lambda_{P,t-1}\})] \qquad (35)$$

With 3 age groups there are 2 parameters, a_t and b_t, that are constrained by only one equation, i.e. Eq(35). As a result, a valid model may arise in many ways, and a_t and b_t are not fully determined by the Intrinsic Linkage constraint. That flexibility in the model makes it possible to have a value of w_t that is less than -1 or greater than 1, as long as the sum in Eq(2) always converges.

Three points should be made. First, if $b_t = 0$, then we again have the case of Eq(29). Hence Eq(31) (or Eq(32) if w is constant) provides a sufficient condition for a valid model, though it is no longer necessary.

Second, if w is constant over time, a long term valid Intrinsic Linkage model again requires that $-1 < w < 0$. To see why, let us consider the model at time t, and assume that $\lambda_{P,t-1}$ and $\lambda_{P,t-2}$ are known. If $\lambda_{P,t-1} = \lambda_{P,t-2}$, then the population is stable (or $\mathbf{A_{t-1}}$ is not primitive). If not, let us assume that $\lambda_{P,t-2} > \lambda_{P,t-1}$ (though the same line of argument holds if $\lambda_{P,t-2} < \lambda_{P,t-1}$). From Eq(34), for $w>0$, a valid model requires that the second term on the right be negative, hence λ_t must be between $\lambda_{P,t-1}$ and $\lambda_{P,t-2}$ and greater than the geometric mean of $\lambda_{P,t-1}$ and $\lambda_{P,t-2}$. With $w>0$, Eq(1) implies that λ_{Pt} must be between λ_t and $\lambda_{P,t-1}$. Now we are back to the starting scenario, but the gap between $\lambda_{P,t-1}$ and λ_{Pt} is substantially smaller than the gap that existed between $\lambda_{P,t-1}$ and $\lambda_{P,t-2}$. Over time, with $w>0$, that gap must go to zero, and the population will either become stable or the population projection matrix will no longer be primitive. Since $w \geq 0$ does not yield a valid long term model, we are left with $-1 < w < 0$.

Third, because the 3-age group model has an additional degree of freedom, we can introduce an additional constraint, say one related to the Net Reproduction Rate (NRR). Following Lotka, the NRR can be thought of as the growth that occurs over a generation, as in the stable population relationship

$$NRR = \exp(rT) \tag{36}$$

where r is Lotka's intrinsic growth rate and T is the length of a generation (Schoen 2006:11). A 3-age group model typically has the reproductive age groups 0-14, 15-29, and 30-44, and age 30 is often close to the length of a generation. In the discrete case, with λ_t representing 15 years of growth from time t−1 to time t, we can write

$$\mathrm{NRR}_t = \lambda_t^2 = (1-a_t-b_t)\,\lambda_t + a_t\,\lambda_t^2 + b_t\,\lambda_t^3 \qquad (37)$$

Using Eqs(37), (1), and (27), we find that the parameters of $\mathbf{A}_t$ are given by

$$a_t = \{\,(-w)\,\lambda_{P,t-1}\,\lambda_{P,t-2}\,(\lambda_t + 1)[\,\lambda_{P,t-1} - \lambda_t] - \lambda_t\,\lambda_{P,t-1}\,\lambda_{P,t-2}\,\}\,/\mathrm{DENOM}$$

and

$$b_t = \{\,\lambda_{P,t-2}\,[w\,\lambda_{P,t-1} + \lambda_t\,][\,\lambda_{P,t-1} - \lambda_t\,]\,\}\,/\,\mathrm{DENOM}$$

where

$$\mathrm{DENOM} = \lambda_t^2\,[\lambda_{P,t-2}\,(\lambda_{P,t-1} - 1) - \lambda_t\,(\lambda_{P,t-2} - 1)\,] \qquad (38)$$

An analyst can thus start with a fixed w and a sequence of NRRs, and derive the accompanying sequences of λ_t's, λ_{Pt}'s, and Leslie matrices.

Specifying the Intrinsic Linkage Leslie matrix for 4 or more age groups.

Leslie matrices with 4 or more age groups have relationships that parallel those in 3-age models. Eqs(31) and (32) still provide sufficient conditions for a valid Leslie matrix in the general and constant w cases, respectively. In equations that parallel Eq(35), there are additional terms on the right side, increasing the likelihood that a valid Leslie matrix exists. For example, in the 4-age case, there is an additional term of the form

$$[\lambda_t^3 - \lambda_{P,t-1}\,\lambda_{P,t-2}\,\lambda_{P,t-3}\,]\,/\,[\lambda_{P,t-1}\,\lambda_{P,t-2}\,\lambda_{P,t-3}\,(\lambda_t - \lambda_{P,t-1})]$$

Still, in the constant w case, the same dynamic discussed after Eq(35) continues to operate, so in long term models the value of w must satisfy $-1 < w < 0$.

NUMERICAL ILLUSTRATIONS INVOLVING INTRINSIC LINKAGES

Here, numerical values for 3-age group Intrinsic Linkage models are presented for (i) a metastable model, (ii) a 3-cycle cyclically stationary model, and (iii) a 4-cycle cyclically stable model calculated from NRRs. Models with 3 reproductive age groups are used to simplify the presentation while preserving the essential population dynamics.

Values in a 3-age group metastable model.

The metastable model allows fertility to steadily increase or decrease, and can provide an analytical bridge between stable population regimes, such as in the transition to stationarity. **Table 1** shows numerical relationships under two values of the constant Intrinsic Linkage parameter w, specifically −0.6 and −0.4. Values for the metastable parameters are c=1.02 and k=1.005, with Leslie matrix value b set at 0.05.

Since the metastable population is growing, fertility values in the first row of the Leslie matrix increase over time, while the relative size of the population in the older age groups decreases. Leslie growth rate λ_t increases over time, as does $\lambda_{Pt}=\lambda_t\lambda_s$. When w increases from −0.6 to −0.4, there is a slight decrease in metastable growth factor λ_s, but over a 10 interval period the resultant difference in births is only that between 1.634 and 1.627, a mere 0.4%. The change in w does have a substantial effect on age-specific fertility values, with the larger w associated with a much earlier fertility schedule. However, because the value of b was fixed, at the same time point the first row of both Leslie matrices had identical third elements.

Values in a 3-age group, 3-cycle cyclically stationary model.

A cyclically stationary population provides analysts with a flexible model for studying fluctuations in the absence of long term growth. The Intrinsic Linkage approach provides explicit relationships that facilitate the construction of such models.

Table 2 shows values for a 3-cycle, cyclically stationary, Intrinsic Linkage model with 3 age groups. For stationarity, we must have $\lambda_{P1}\ \lambda_{P2}\ \lambda_{P3} = 1$, i.e. no growth over each cycle. Combined with Eq(20), the stationarity constraint enables the model to be solved directly for the λ_{Pt} and constant w, though the λ_t must be chosen carefully to allow a valid model. An easier approach is to choose a value of w in the broad range that satisfies the sufficiency condition in Eqs(30) or (32), and solve for the λ_{Pt} and one of the λ_t. Table 2 does the latter, with $\lambda_1 = 0.9$, $\lambda_2 = 1.1$, and w set at either −0.8 or −0.4. In the Leslie matrix, there is only one constraint on each pair of a_j and b_j. To simplify matters while insuring a valid model, b_j is always set at 0.02.

Table 2 indicates that the choice of w has little effect on the calculated value of λ_3, but a definite impact on the values of the λ_{Pt}. The more negative value of w leads to large values for the a_j, and a concentration of fertility in the second age group. The less negative value of w is associated with a marked shift in fertility to the first age group. Because the cycle length is the same as the number of age groups, total population size is constant over time, and equals 3.0666 when $w=-0.8$ and 2.9185 when $w=-0.4$. Population vector $\mathbf{x_3}$ is identical to $\mathbf{x_0}$, and Leslie matrix $\mathbf{A_4}$ is the same as $\mathbf{A_1}$. Within a cycle, however, cohort size varies by up to 24% when $w=-0.8$ and up to 19% when $w=-0.4$. For both choices of w, the sum of the three λ_{Pt} equals the sum of the three λ_t.

Values in a 3-age group, 4-cycle cyclically stable model derived from NRRs.

Cyclical growth (or decline) is common, and cycles four intervals long are often associated with seasonal variations. **Table 3** presents values from a 3-age group, 4-cycle constant w cyclically stable population where the growth rates are obtained from NRRs using Eq(37). Because the NRRs add a further constraint, the sufficiency condition in Eq(32) no longer applies, and a valid model is not always possible. With the NRR values used here, i.e. 0.95, 1.05, 1.25, and 0.85, no valid model was found when w was −0.8 or less or −0.5 or more.

Table 3 shows Intrinsically Linked model values when w is −0.7 and −0.6. When $w=-0.7$, the λ_P values are consistently further from 1 than when $w=-0.6$. Moreover, when $w=-0.7$, the a_j values are uniformly bigger and the b_j values uniformly smaller than when $w=-0.6$, signaling that a more negative w shifts fertility away from the highest age and toward the second age group. The effect is most dramatic in Leslie matrix $\mathbf{A_3}$, where the fertility value in the second age group when $w=-0.6$ is quite small.

The cyclically stable population declines in size from cycle to cycle, as the product of the four λ_j is 0.93353 when $w=-0.7$ and 0.97872 when $w=-0.6$. Population vector $\mathbf{x_4}$ is identical to $\mathbf{x_0}$ times the product of the four λ_j's. Between times 0 and 4, total population size varies from 3.26 to 3.63 when $w=-0.7$ and from 3.19 to 3.44 when $w=-0.6$.

There is substantial variation in cohort size within each cycle. When $w=-0.7$, relative cohort sizes vary by over 45%, while when $w=-0.6$, they vary by 33%. Cohort NRRs vary even more, going from 0.49 to 1.66 when $w=-0.7$ and from 0.34 to 1.83 when $w=-0.6$. The NRRs of successive cohorts alternate between being substantially above replacement to being substantially below replacement.

SUMMARY AND CONCLUSIONS

The Intrinsic Linkage relationship in Eq(1) connects λ_t, the intrinsic growth rate of the time t Leslie matrix, to λ_{Pt} and $\lambda_{P,t-1}$, the model growth rate of births at times t and t−1. The Intrinsic Linkage approach is built on the idea that it is plausible to represent λ_t as a linear combination of λ_{Pt} and $\lambda_{P,t-1}$. With Intrinsic Linkage, each λ_{Pt} can be expressed as a convergent power series in λ_t and the linear weight parameter, w_t.

When λ_t has a polynomial, exponential, or cyclical trajectory and w is constant over time, new closed form relationships between λ_t and λ_{Pt} emerge. The construction of Intrinsically Linked models is straightforward, and several illustrative numerical examples of cyclical and metastable models are presented.

Much of the demographic content of Leslie matrix $\mathbf{A_t}$, of whatever size, appears to be embodied in a single number: eigenvalue λ_t. The dominant root of $\mathbf{A_t}$ reflects not only intrinsic growth but, in the absence of mortality before the end of childbearing, the long term age composition as well. In determining the size and composition of the model population, much of the contribution of all the subordinate components of $\mathbf{A_t}$ can be captured by Intrinsic Linkage parameter w. The analytical advantages of the Intrinsic Linkage approach flow from its focus on those two key measures.

In sum, the Intrinsic Linkage approach provides a new and flexible way to link population projection matrices with the age structure of observed (or model) populations. Intrinsic Linkage extends current methods for the dynamic modeling of birth-death models, including cyclically stable and cyclically stationary models, and affords new options for analyzing populations with changing vital rates.

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Table 1. Numerical relationships in a 3-reproductive age group birth-death metastable model with Intrinsic Linkage, for 2 values of constant parameter w

Item	$w = -0.6$	$w = -0.4$
a	0.41747	0.22790
λ_s	1.00187	1.00142
λ_1	1.02510	1.02510
λ_{P1}	1.02702	1.02656
λ_{10}	1.07216	1.07216
λ_{P10}	1.07417	1.07369
First row of Leslie matrix $\mathbf{A}_1$	[.54589, .43869, .05386]	[.74023, .23948, .05386]
First row of Leslie matrix $\mathbf{A}_{10}$	[.57095, .47990, .06162]	[.77421, .26198, .06162]
Time 0 model population $\mathbf{x}_0'$	[1, .97856, .96237]	[1, .97900, .96323]
Time 10 model population $\mathbf{x}_{10}'$	(1.63397) [1, .93561, .87974]	(1.62672) [1, .93603, .88052]

NOTES. The model is based on text Eqs(12)-(14) with c=1.02, k=1.005, and b=0.05. Population growth over time was accumulated using the relationship that the product of k^j, j going from 1 to n equals $k^{n(n+1)/2}$. The value of parameter a in the 3-age metastable model is found from

$$a = \{b\, k\, (k-w)^3 - (1-w)^2\, [w\, (1-k)+b\, (k-z)]\} \,/\, \{w^2\, (1-k) - k^2\, (1-w) + (k-w)\}$$

Table 2. Numerical relationships in a Cyclically Stationary 3-reproductive age group, 3-cycle birth-death model with Intrinsic Linkage, for 2 values of constant parameter w

Item	$w = -0.8$	$w = -0.4$
λ_3	1.03749	1.02517
λ_{P1}	0.92144	0.87623
λ_{P2}	1.24285	1.18951
λ_{P3}	0.87320	0.95944
a_1	.94148	0.33271
a_2	.61814	0.28422
a_3	.95108	0.46533
b_1	.02	.02
b_2	.02	.02
b_3	.02	.02
First row of Leslie matrix $\mathbf{A_1}$	[.03467, .76260, .01458]	[.58256, .26949, .01458]
First row of Leslie matrix $\mathbf{A_2}$	[.39805, .74794, .02662]	[.76535, .34391, .02662]
First row of Leslie matrix $\mathbf{A_3}$	[.03001, 1.02373, .02233]	[.52763, .48905, .02155]
Time 0 model population $\mathbf{x_0}'$	[1, 1.14521, .92144]	[1, 1.04228, .87623]
Time 1 model population $\mathbf{x_1}'$	(.92144)[1, 1.08526, 1.24285]	(.87623) [1, 1.14126, 1.18951]
Time 2 model population $\mathbf{x_2}'$	(1.14521)[1, .80460, .87320]	(1.04228) [1, .84068, .95944]

NOTES. The model is based on text Eqs(19)-(22) with λ_1=0.9 and λ_2=1.1. Since $\lambda_{P1}\ \lambda_{P2}\ \lambda_{P3}$ =1, there is no population growth over each 3-interval cycle.

Table 3. Numerical relationships in a Cyclically Stable 3-reproductive age group, 4-cycle birth-death model with Intrinsic Linkage, for 2 values of constant parameter w

Item	w = −0.7	w = −0.6
λ_1	.97468	.97468
λ_2	1.02470	1.02470
λ_3	1.11803	1.11803
λ_4	.92195	.92195
λ_{P1}	1.17601	1.10784
λ_{P2}	.91877	.97481
λ_{P3}	1.25752	1.20397
λ_{P4}	.68706	.75275
a_1	.42408	.41490
a_2	.90853	.87013
a_3	.35493	.03459
a_4	.90669	.44133
b_1	.29165	.29630
b_2	.04518	.06414
b_3	.30456	.45580
b_4	.04855	.29068
First row of $\mathbf{A_1}$	[.27707, .40288, .27005]	[.28149, .39416, .27436]
First row of $\mathbf{A_2}$	[.04744, .95396, .04861]	[.06735, .91364, .06901]
First row of $\mathbf{A_3}$	[.38070, .44366, .42564]	[.56975, .04324, .63700]
First row of $\mathbf{A_4}$	[.04127, .77069, .03805]	[.24707, .37513, .22779]
Population $\mathbf{x_0}'$	[1, 1.45548, 1.15742]	[1, 1.32847, 1.10341]
Population $\mathbf{x_1}'$	(1.17601)[1, .85033, 1.23764]	(1.10784)[1, .90266, 1.19915]
Population $\mathbf{x_2}'$	(1.08049)[1, 1.08841, .92551]	(1.07993)[1, 1.02584, .92598]
Population $\mathbf{x_3}'$	(1.35873)[1, .79522, .86552]	(1.30020)[1, .83059, .85205]

NOTES. The model is based on text Eqs(19)-(22) and (37) with the number in the first age group at time 0 scaled to 1 and NRR_1=0.95, NRR_2=1.05, NRR_3=1.25, and NRR_4=0.85. There is population decline over each 4-interval cycle, by a factor of 0.93353 when w=−0.7 and by a factor of 0.97872 when w=−0.6.