Submission to PAA 2012

Estimating Demographic Parameters with Uncertainty from Fragmentary Data *

Mark C. Wheldon, Adrian E. Raftery, Samuel J. Clark
University of Washington

Patrick Gerland
United Nations Population Division

September 2, 2011

ABSTRACT

Current methods for reconstructing human population structures of the past are deterministic or do not formally account for measurement error. We propose a method for simultaneously estimating age-specific population counts, fertility rates, mortality rates and net international migration flows from fragmentary data, that incorporates measurement error. Inference is based on joint posterior probability distributions which yield fully probabilistic interval estimates. It is designed for the kind of data

---

[*] Mark C. Wheldon is a PhD. candidate in the Department of Statistics, University of Washington, Seattle, WA 98195 (e-mail: mwheldon@uw.edu); Adrian E. Raftery is Professor of Statistics and Sociology at the University of Washington (e-mail: raftery@u.washington.edu); Samuel J. Clark is an Assistant Professor in the Department of Sociology at the University of Washington and Researcher at the Institute of Behavioral Science (IBS), University of Colorado at Boulder, CO, USA and MRC/Wits Rural Public Health and Health Transitions Research Unit (Agincourt), School of Public Health, Faculty of Health Sciences, University of the Witwatersrand, Johannesburg, South Africa (e-mail: samclark@u.washington.edu); and Patrick Gerland is a Population Affairs Officer with the United Nations Department of Economic and Social Affairs, Population Division, New York, NY 10017 (e-mail: gerland@un.org). This work was supported by grants R01 HD054511 and K01 HD057246 from the Eunice Kennedy Shriver National Institute of Child Health and Human Development. The views and opinions expressed in this paper are those of the authors and do not necessarily represent those of the United Nations. This paper has not been formally edited and cleared by the United Nations.

commonly collected in modern demographic surveys and censuses. Population dynamics over the period of reconstruction are modeled by embedding formal demographic accounting relationships in a Bayesian hierarchical model. Informative priors are specified for vital rates, migration rates, population counts at baseline, and the accuracies of their respective measurements. We investigate calibration of central posterior marginal probability intervals by simulation and demonstrate the method by reconstructing the female population of Burkina Faso from 1960 to 2000.

Keywords

Bayesian hierarchical model; Cohort component model; Fertility rate; Markov chain Monte Carlo; International migration; Mortality rate.

1. INTRODUCTION

Every two years the United Nations Population Division (UNPD) publishes detailed estimates of key demographic parameters for all countries in the world; they appear in the *United Nations World Population Prospects* (UNWPP; e.g., United Nations 2009). The parameters reported include age-specific fertility and mortality rates (vital rates), population counts and international migration rates. These figures are used for the development and assessment of policy and are of particular importance for countries that lack their own well-resourced official statistical systems. For many of these countries, the UNPD is a key partner in the process of compiling, analyzing and publishing the data.

The *United Nations World Population Prospects* (UNWPP) tables can be classified into two broad groups, known as estimates and projections. Projections are predictions about demographic parameters in the future, while estimates concern population structures of the past. We will use the term reconstruction which we find less ambiguous. Nevertheless, use of the term “estimate” agrees with standard usage in statistics; data about demographic parameters in the past are used to estimate the true values. Between the biennial editions of the UNWPP, new data are collected, and old data become available, thus revisions must be made to the reconstructions to accommodate it.

Implicit in the previous paragraph is the assertion that there is uncertainty in the measurements. This is hardly controversial; demographers have long acknowledged that some parameters are almost immune to perfect measurement and many so-called indirect methods exist for estimating them (e.g., United Nations 1983). Most of these, however, are focused on obtaining a best point estimate of a single quantity. Although it has been standard practice to produce ranges of estimates for many parameters based on different scenarios, high, medium and low fertility for example, the ranges produced cannot be interpreted probabilistically. This approach is still commonly used for projections, but since the mid 1990s new methods began to appear which do yield probabilistic predictions; Lee (1998) and Booth (2006) provided reviews. In contrast, there has been relatively little work on the development of fully probabilistic methods for demographic reconstruction. Here we propose a general method for reconstruction that accounts for measurement uncertainty and works with the type of demographic data that have commonly been collected for most countries over the last 60 years or so. It has been designed to fit within the UNPD’s existing work-flow, but we hope it is general enough to have broad appeal.

In Section 2 we will briefly describe existing reconstruction techniques from the demographic literature; a discussion of relevant work in other areas such as ecology and fisheries will be deferred until the Discussion. In Section 3 we lay down our notation and describe the method. In Section 4, we investigate some statistical properties of our method through simulation before applying it to real data from Burkina Faso in Section 5. We close with a summary of the results, a literature review and a discussion of some technical issues in Section 6.

2. METHODS OF RECONSTRUCTION IN DEMOGRAPHY

Outside of official statistical agencies, demographers have undertaken population reconstructions for at least two reasons: historical studies of populations of the past (e.g., Wrigley and Schofield 1981) and to estimate excess mortality in crises such as famine or social upheaval (e.g., Boyle and Grada 1986, Heuveline 1998, Merli 1998, Goodkind and West 2001). The most commonly used methods are based on the demographic balancing equations. These are the basic accounting relationships which state that the population size at time $t + \delta$ is equal to the size at time t plus births and immigrants, minus deaths and emigrants. These relationships are encoded in the cohort component model of population projection (CCMPP; Lewis 1942, Leslie 1945, 1948). Given the size and age-structure of a population at some baseline time, its size and structure at any point in the future can be determined from the baseline population and the fertility, mortality and international migration rates that prevail over the period of reconstruction.

The back projection method of reconstruction (Wrigley and Schofield 1981) attempts to apply these relationships in reverse by using an estimate of population structure at the terminal year of the period of reconstruction. This approach is problematic since the cohort component model of population projection (CCMPP) procedure is not formally invertible. To produce sensible results, some additional constraints have to be imposed or somewhat ad-hoc fixes applied.

In response to these concerns, Lee (1971, 1974) proposed the method of inverse projection. Inverse projection enacts the reconstruction forward through time; it is named for the fact that, instead of estimating counts of births and deaths from rates, count data are used to infer rates.

Both of these methods have been developed since their inception. For example, McCaa and Barbi (2004) and Rosina (2004) described extensions of inverse projection. Oeppen (1993) proposed generalize inverse projection (GIP). This method produces estimates of demographic parameters by minimizing the discrepancy between population structures implied by the estimates and those observed. Additional constraints, such as a smoothness criterion on estimates of migration, are used to ensure parameter identifiability. However, all of the extended methods remain purely deterministic.

A stochastic reconstruction method was proposed by Bertino and Sonnino (2003) who modeled childbirth and death as inhomogeneous Poisson processes. The method is designed to work with counts of births and deaths aggregated over age and so requires the analyst to specify model age-patterns for fertility and mortality, although different schedules can be chosen for different sub-periods. These schedules are taken as the intensity functions of the process and realizations are simulated through time to produce sequences of empirical estimates of population age structures. Hence, the only source of variation accounted for is natural variation around the demographic rates; the total numbers of deaths by year are assumed to be recorded without error. Moreover, international migration is assumed to be negligible over the period of reconstruction.

Finally, all of the above methods were designed to work with aggregate data of the kind sometimes found in European parish registers. The data available for many countries in our intended application is not as detailed, even for periods stretching as little as 50 years into the past. The aim of this article is to introduce a new method of reconstruction that uses measurements of demographic parameters available for 40–50 years in the past from multiple noisy data sources. Moreover, uncertainty due to measurement error is expressed as probability distributions and intervals rather than deterministic scenario-based ranges.

3. METHOD

In this article we restrict our attention to the dynamics of populations of females only; extension to two-sex populations is left for future work. The parameters of interest are the basic demographic parameters: age- and time-specific vital rates, net international migration flows and population counts. In the sequel we will refer to "international migration" as simply "migration" as this is the only type we consider.

The method we propose can be viewed as a reconciliation of two different estimates of age-specific population counts, namely the estimate based on direct enumeration or surveys (e.g., censuses) and the estimate from applying the CCMPP to measurements of fertility, mortality and migration. The reconciliation is through a Bayesian model which provides probabilistic posterior distributions of all of the parameters of interest. The posterior distributions summarize uncertainty due to measurement error. Probability intervals for summary measures such as total fertility rate (TFR) and life expectancy at birth (LEB) can also be easily calculated.

3.1. Notation and Terminology

We will index all demographic parameters by age, denoted by a, and time, denoted by t. Age is measured on an open-ended scale with the largest value A_+, 80 years and older, for example. We will be interested in the value of the key parameters at five-year increments of age and time, $a = 0, 5, \ldots, A_+$ and $t = t_0, t_0 + 5, \ldots, T$. We will use the symbol A_{++} to refer to the age at time t of those who were aged A_+ for the first time at time $t - 5$. Furthermore, $a_L^{[\text{fert}]}$ and $a_U^{[\text{fert}]}$ will refer to the lower and upper ages of non-zero fertility, respectively. Finally, we will need to refer to a special subset of time-points for which censuses or other direct estimates of the population counts are available; we denote these times by $t_L^{[\text{cen}]}, \ldots, t_U^{[\text{cen}]}$ where "L" and "U" indicate lower and upper limits.

We adopt the standard demographic definition of fertility rate, and let $f_{a,t}$ denote the ratio of the number of births to women in the age range $[a, a+5)$ to the number of person years lived by those women over the time period $[t, t+5)$ (Preston, Heuveline, and Guillot 2001).

Mortality will be expressed as the proportion of people who survive from one age group to the next. Thus we set $s_{a,t}$ to be the proportion in the age range $[a-5, a)$ at time t who survive to be aged $[a, a+5)$ at time $t+5$. That is, $s_{a,t}$ is the proportion of people surviving *into* the age range $[a, a+5)$ over the 5 years between t and $t+5$. The number of people aged $[a, a+5)$ at exact time t will be denoted by $n_{a,t}$.

Net migration over a period of time will be measured in proportionate terms; let $g_{a,t}$ be the net number of migrants to the population aged $[a, a+5)$ who migrate during the time period $[t, t+5)$, expressed as a proportion of $n_{a,t}$. Thus, the share of $n_{a,t+5} - n_{a,t}$ attributable to migration is $g_{a,t} n_{a,t}$.

Bold face and omission of the age index will be used to denote vectors of particular age-specific parameters for a given time. For example, $\boldsymbol{f}_t$ is the vector of age-specific fertility rates for the period $[t, t+5)$. When convenient, we will use $\boldsymbol{\theta}$ to represent

$$(\boldsymbol{n}_{t=t_0}, \boldsymbol{f}_{t_0}, \boldsymbol{f}_{t_0+5}, \ldots, \boldsymbol{f}_T, \boldsymbol{s}_{t_0}, \ldots, \boldsymbol{s}_T, \boldsymbol{g}_{t_0}, \ldots, \boldsymbol{g}_T),$$

the vector of all age-time-specific parameters.

To avoid confusion when we refer to periods of time, we will use the term "reconstruction period" to indicate the span of time over which the population is reconstructed.

3.2. Model Description

3.2.1. A Hierarchical Model

Given $\boldsymbol{n}_t$, $\boldsymbol{f}_t$, $\boldsymbol{s}_t$ and $\boldsymbol{g}_t$, the CCMPP gives $\boldsymbol{n}_{t+5}$ via the deterministic relationship

$$\begin{bmatrix} n_{0,t+5} \\ n_{5,t+5} \\ \vdots \\ n_{A_+-5,t+5} \\ n_{A_+,t+5} \end{bmatrix} = \begin{bmatrix} \widetilde{f}_{0,t} & \widetilde{f}_{5,t} & \cdots & \widetilde{f}_{A_+-5,t} & \widetilde{f}_{A_+,t} \\ s_{5,t} & 0 & & 0 & 0 \\ 0 & s_{10,t} & \ddots & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & \cdots & s_{A_+,t} & s_{A_{++},t} \end{bmatrix} \begin{bmatrix} n_{0,t}(1+g_{0,t}/2) \\ n_{5,t}(1+g_{0,t}/2) \\ \vdots \\ n_{A_+-5,t}(1+g_{A_+-5,t}/2) \\ n_{A_+,t}(1+g_{A_+,t}/2) \end{bmatrix} + \begin{bmatrix} n_{0,t}g_{0,t}/2 \\ n_{5,t}g_{5,t}/2 \\ \vdots \\ n_{A_+-5,t}g_{A_+-5,t}/2 \\ n_{A_+,t}g_{A_+,t}/2 \end{bmatrix} \tag{1}$$

where

$$\widetilde{f}_{a,t} = 5(s_{0,t}/2)(1+SRB)^{-1}\big(f_{a,t} + f_{(a+5),t} \cdot s_{a+5,t}\big),$$

and the sex ratio at birth (SRB) is taken to be fixed at 1.05, a standard default (Preston et al. 2001). We use $s_{A_{++},t}$ to denote the proportion of those in the open-ended interval who survive to time $t+5$. We will abbreviate (1) as $\boldsymbol{n}_{t+5} = M(\boldsymbol{n}_t, \boldsymbol{f}_t, \boldsymbol{s}_t, \boldsymbol{g}_t)$.

We use an asterisk (*) to distinguish between the unknown, true values and measurements of them based on data. Specifically, let $f_{a,t}$ be the true fertility rate and $f^*_{a,t}$ be a measurement of it, and similarly for all other parameters of interest. In what follows, we will use $\mathcal{D}$ to represent the set of all such measurements.

In addition to a measurement of the population size at baseline ($\boldsymbol{n}^*_{t=t_0}$), measurements of population counts at exact times in $(t_0, T]$ are typically available. These direct measurements might be from censuses or demographic surveys. Therefore, if $\boldsymbol{n}^*_{t_0}$, $\boldsymbol{n}^*_{t+5}$, $\boldsymbol{f}^*_t$, $\boldsymbol{s}^*_t$ and $\boldsymbol{g}^*_t$ are available, we have two estimates of $\boldsymbol{n}_{t+5}$, namely $\boldsymbol{n}^*_{t+5}$ and $M(\boldsymbol{n}_t = M(\boldsymbol{n}_{t-5}), \boldsymbol{f}^*_t, \boldsymbol{s}^*_t, \boldsymbol{g}^*_t)$ where $M(\boldsymbol{n}_{t_0})$ is defined to be $\boldsymbol{n}^*_{t_0}$. We combine these two sources into a joint distribution of all parameters in the following four-level hierarchical model, where we have made the simplifying assumption that the elements of $\boldsymbol{\theta}$ are mutually independent *a priori*:

$$\text{Level 1}: \quad \log n^*_{a,t} \,|\, n_{a,t}, \sigma^2_n \sim \text{Normal}\left(\log n_{a,t}, \sigma^2_n\right), \; t = t^{[\text{cen}]}_L, \ldots, t^{[\text{cen}]}_U \tag{2}$$

$$\text{Level 2}: \quad n_{a,t} \,|\, \boldsymbol{n}_{t-5}, \boldsymbol{f}_{t-5}, \boldsymbol{s}_{t-5}, \boldsymbol{g}_{t-5} = M(\boldsymbol{n}_{t-5}, \boldsymbol{f}_{t-5}, \boldsymbol{s}_{t-5}, \boldsymbol{g}_{t-5}), \quad t = t_0 + 5, t_0 + 10, \ldots, T \tag{3}$$

$$\text{Level 3}: \quad \log n_{a,t_0} \,|\, \sigma^2_n \sim \text{Normal}\left(\log n^*_{a,t_0}, \sigma^2_n\right), \; a = 0, 5, \ldots, A_+ \tag{4}$$

$$\log f_{a,t} \,|\, \sigma^2_f \sim \begin{cases} \text{Normal}\left(\log f^*_{a,t}, \sigma^2_f\right), & a = a^{[\text{fert}]}_L, \ldots, a^{[\text{fert}]}_U \\ & t = t_0, t_0 + 5, \ldots, T \\ \text{undefined}, & \text{otherwise} \end{cases} \tag{5}$$

$$\text{logit}\, s_{a,t} \,|\, \sigma^2_s \sim \text{Normal}\left(\text{logit}\, s^*_{a,t}, \sigma^2_s\right), \; a = 0, 5, \ldots, A_{++} \quad t = t_0, t_0 + 5, \ldots T \tag{6}$$

$$g_{a,t} \,|\, \sigma^2_g \sim \text{Normal}\left(g^*_{a,t}, \sigma^2_g\right), \; a = 0, 5, \ldots, A_+, \quad t = t_0, t_0 + 5, \ldots, T \tag{7}$$

$$\text{Level 4}: \quad \sigma^2_v \sim \text{InvGamma}(\alpha_v, \beta_v), \; v \in \{n, f, s, g\}. \tag{8}$$

For $0 < x < 1$, $\text{logit}\, x \equiv \log(x/(1 - x))$.

In standard Bayesian terms, expressions (4)–(7) could be viewed as informative priors for $\boldsymbol{\theta}$ while (2) is the likelihood of $n^*_{a,t} \,|\, \boldsymbol{\theta}$. Inference will be based on the joint posterior distribution of these parameters, $p(\boldsymbol{\theta} \,|\, \mathcal{D})$. We also explicitly impose the restriction that all population counts be positive. Therefore, the joint prior on all parameters at time t, $p(\boldsymbol{n}_{t_0}, \boldsymbol{f}_t, \boldsymbol{s}_t, \boldsymbol{g}_t)$, is multiplied by

$$I\big(M(\boldsymbol{n}_t, \boldsymbol{f}_t, \boldsymbol{s}_t, \boldsymbol{g}_t) > 0\big) \equiv \begin{cases} 1; & n_{a,t+5} \geq 0,\ a = 0, \ldots, A_+ \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

The quantities involved, and their dependence relations, are summarized in Figure 1.

3.2.2. Determining the Hyperparameters

To determine plausible values of α_v and β_v, $v \in \{n, f, s, g\}$, we view the σ^2_v as representing the measurements of the respective demographic parameters. Although prior knowledge about these accuracies is unlikely to be exact, we expect that informative estimates can be derived from experts' knowledge of the data sources. Methods for eliciting prior information from experts are numerous (e.g., O'Hagan et al. 2006). Here, we use a straightforward method based on the mean absolute error (MAE) of the transformed measurements.

Taking fertility rate as an example, note that (5) implies $\text{MAE}(\log f_{a,t} \,|\, \sigma^2_f) \equiv \text{E}(|\log f_{a,t} - \log f^*_{a,t}| \,|\, \sigma^2_f) = \sigma_f \sqrt{2/\pi}$. The prior distribution for σ^2_f can be specified by choosing quantiles for $\text{MAE}(\log f_{a,t} \,|\, \sigma^2_f)$. Suppose expert opinion is that $\text{MAE}(\log f_{a,t} \,|\, \sigma^2_f)$ is close to 0.1, but could be as high as 0.5. This suggests setting $\text{median}(\sigma^2_f) = (0.1)^2\pi/2 \approx 0.0151$ and the 0.975 quantile to 0.5. To find an inverse gamma distribution with these quantiles, we fixed α_f at a range of values between 0.3 and 6 and chose β_f such that $\text{median}(\text{MAE}(\log f_{a,t} \,|\, \sigma^2_f)) = 0.1$. The parameter α_f was then chosen such that the 0.95 quantile of $\text{MAE}(\log f_{a,t} \,|\, \sigma^2_f)$ was about 0.5.

Since demographers are more used to thinking about untransformed fertility rates,

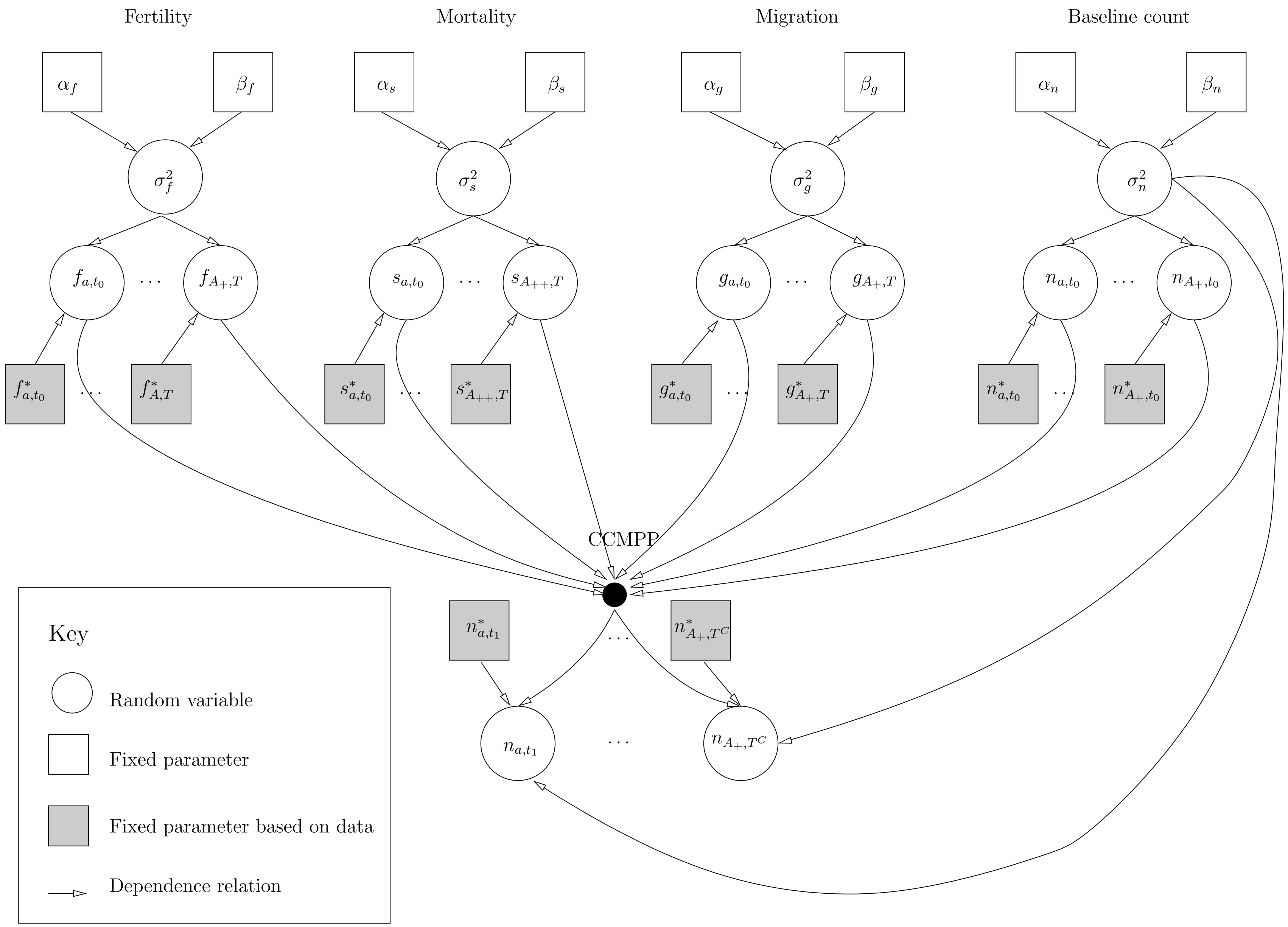


Figure 1. Schematic Showing the Relationships Between Parameters, Measurements and the CCMPP in the Estimation Model.

it is useful to consider what specifying $\mathrm{MAE}(\log f_{a,t} \,|\, \sigma_f^2)$ means on the original scale. On the log scale, MAE approximates mean absolute relative error (MARE) on the original scale where $\mathrm{MARE}(f_{a,t} \,|\, \sigma_f^2) \equiv \mathrm{E}(|f_{a,t} - f_{a,t}^*| \,|\, \sigma_f^2)/f_{a,t}^*$. This approximation is good for the MAE values used here. The MARE has been used previously by demographers as a way of quantifying measurement accuracy (Keilman 1998).

The population count variance, σ_n^2, is also modeled on the log scale and α_n, β_n are found in the same way as α_f and β_f. Survival is measured on the logit scale, but this should not be too difficult to interpret. Note that $\mathrm{E}(|\,\mathrm{logit}\,\bar{s}_{a,t} - \mathrm{logit}\,\bar{s}_{a,t}^*| \,|\, \sigma_s^2) = \mathrm{E}(|\,\mathrm{logit}\,s_{a,t} - \mathrm{logit}\,s_{a,t}^*| \,|\, \sigma_s^2)$ where $\bar{s}_{a,t} = 1 - s_{a,t}$. In practice, the $\bar{s}_{a,t}$ are close to zero, so that $\bar{s}_{a,t} \approx \bar{s}_{a,t}/(1 - \bar{s}_{a,t})$. Thus specifying the MAE of the log-odds of survival is not that different from specifying it for the log probability of death. For the migration parameter, which is a proportion, we specify the MAE directly.

3.2.3. Summary of Model

We model the transformed vital and migration rates as unimodal distributions centered at the relevant measurements which serve as fixed prior means. This is equivalent to the assumption that the measurements are median-unbiased but noisy, which might require preprocessing of the vital rate data. Techniques for estimating, and hence removing, the bias in vital rate estimates are probably best developed separately as they will depend on data collection methods and other data-specific factors. For example, building on work by Hill et al. (1999) and UNICEF et al. (2007), Alkema et al. (2008) developed a technique for removing the bias in TFR measurements.

The use of informative priors for both the mean and the variance parameters in (4)–(7) is necessary because the model will be technically over-parameterized and hence under-identified in most practical applications.

3.3. Estimation

Samples from the joint posterior can be drawn using a Markov chain Monte Carlo (MCMC) sampler (Metropolis et al. 1953, Hastings 1970, Geman and Geman 1984). Without the restriction in (9), the full conditional posterior distributions for the variance hyperparameters would be the usual conjugate inverse gamma distributions. With the restriction, the conjugate forms are not exactly correct but will probably be very close to the true full conditionals. Therefore, to update these parameters we use the conjugate full conditional distributions as proposal densities in Metropolis-Hastings (M-H) steps. The posterior densities of the remaining parameters are not easy to express analytically since each vital rate enters the likelihood through the map M. Therefore, these parameters are updated using M-H steps with univariate normal proposal densities, with variances tuned by the method of Raftery and Lewis (1996). We will use the term “iteration” to refer to one complete sweep through all age-time specific parameters and variance parameters.

The *R* environment for statistical computing (R Development Core Team 2010), together with the *CODA* package (Plummer et al. 2006, 2010) were used for all data manipulation, model estimation and output analysis.

4. SIMULATION STUDY

Simulation was used to investigate whether the true vital rates, migration proportions and population counts could be recovered from noisy data under the model. To this end, we used a small, synthetic population for which the true values of all key parameters were known. The estimation procedure was applied to noisy measurements of the true parameter values and central posterior marginal probability intervals were compared to the truth. Calibration of the intervals was assessed by replication.

4.1. Method

4.1.1. Inputs

The true vital and migration rates assumed to have prevailed in this population are shown in Table 1. This is not intended to be a realistic model of a human population; datasets typically encountered in human demography have up to 18 age categories and any number of time periods. However, we believe that this reduced population model is of sufficient size and complexity to explore the characteristics of the statistical model while not being too computationally expensive. The TFR and LEB were kept constant at 0.7 births per woman per year and 15.61 years, respectively, for the duration of the reconstruction. A varying pattern of migration was chosen which consisted of net out-migration in the first half of the reconstruction period followed by net in-migration in the second half. The magnitude of the flows was quite volatile, varying from 13 percent to 26 percent of the receiving population. Moreover, migration in both directions was concentrated in the two middle age groups.

The true population counts are shown in Table 2. Those for 1960 were chosen to represent a young population. Those for the subsequent time periods were derived by applying the CCMPP to the 1960 population using the vital rate and migration parameters in Table 1. Therefore, the underlying true population dynamics over the reconstruction period were completely and deterministically defined by (1).

4.1.2. Study Design

The coverage of central marginal posterior probability intervals under the model was estimated by the following experiment. For $j = 1, \ldots, J$:

1. Randomly sample $\sigma_v^{2[j]}$, $v = n, f, s, g$ from (8).

2. Generate measurements $f_{a,t}^{*[j]}, g_{a,t}^{*[j]}$ for $a = 0, \ldots, 15+$, $t = 1960, \ldots, 1975$, $s_{a,t}^{*[j]}$,

Table 1. True Vital Rates Used in the Simulation Study.

	Time period			
a	[1960, 1965)	[1965, 1970)	[1970, 1975)	[1975, 1980)
		Fertility Rate		
0	0.00	0.00	0.00	0.00
5	0.40	0.40	0.40	0.40
10	0.30	0.30	0.30	0.30
15	0.00	0.00	0.00	0.00
		Survival Proportion		
0	0.90	0.90	0.90	0.90
5	0.95	0.95	0.95	0.95
10	0.85	0.85	0.85	0.85
15	0.80	0.80	0.80	0.80
20+	0.10	0.10	0.10	0.10
		Migration Proportion		
0	-0.03	-0.05	0.03	0.05
5	-0.05	-0.10	0.05	0.10
10	-0.06	-0.11	0.06	0.11
15	-0.01	-0.01	0.01	0.01

Table 2. True Population Counts Used in Simulation Study.

	Year				
a	1960	1965	1970	1975	1980
0	7500	8482	9453	11436	14504
5	6000	6886	7512	9280	11600
10	4000	4862	5293	6690	8651
15	3000	3404	3998	4762	6149

Table 3. Hyperparameter Details. Values of the σ_v^2 and selected implied quantiles of the mean absolute error (MAE) of the respective demographic parameters, used in the simulation study and the application to Burkina Faso. For $v \in \{n, f\}$, $\text{MAE}(\log v_{a,t} \,|\, \sigma_v^2) = \sigma_v \sqrt{2/\pi}$; $\text{MAE}(\text{logit}\, s_{a,t} \,|\, \sigma_s^2) = \sigma_s \sqrt{2/\pi}$; $\text{MAE}(g_{a,t} \,|\, \sigma_g^2) = \sigma_g \sqrt{2/\pi}$.

		MAE quantiles				
α	β	0.025	0.25	0.5	0.75	0.975
1	0.0109	0.0433	0.0707	0.1	0.1552	0.5232
1	0.0436	0.0867	0.1414	0.2	0.3104	1.0465

for $a = 0, \ldots, 20+$ $t = 1960, \ldots, 1975$, and $n^{*[j]}_{a,t=t_0}$ for $a = 0, \ldots, 15+$ from the distributions in (4)–(7).

3. Generate measurements $n^{*[j]}_{a,t}$ for $a = 0, \ldots, 15+$ and t in $t = 1965, \ldots, 1975$ from (2).
4. Check that (9) is satisfied by the measurements; if not return to step 1.
5. Draw a large MCMC sample from the joint posterior and find the 0.025, 0.5 and 0.975 quantiles of the marginal distribution of each parameter.

The estimated coverage is then the proportion of the J central posterior marginal intervals containing the known, true value for each parameter.

The hyperparameters of the inverse gamma distributions were determined as described in Section 3.2.2. The median MAEs for log fertility rate, logit survival and log population counts were set to 0.1 with 0.975 quantiles of approximately 0.5. The same figures for migration were set to 0.2 and approximately 1, respectively. The resulting values of α_v, β_v and MAE quantiles are shown in Table 3.

We set $J = 200$ and applied the estimation method described in Section 3.3. Initial values for the population counts, vital rates and migration proportions were set to the measured values. Initial values for the variances were arbitrarily set to five as this appeared to have a negligible effect on the final results. Approximately 9000

Table 4. Estimated Coverage Probabilities of 95 Percent Posterior Intervals for all Demographic Parameters of Interest.

years	ages	population	fertility	survival	migration
1960	[0, 5)	0.90			
	[5, 10)	0.94			
	[10, 15)	0.89			
	[15, 20)	0.88			
[1960, 1965)	[0, 5)			0.94	0.94
	[5, 10)		0.93	0.97	0.94
	[10, 15)		0.95	0.95	0.93
	20+			0.94	
[1965, 1970)	[0, 5)			0.93	0.92
	[5, 10)		0.94	0.93	0.94
	[10, 15)		0.96	0.92	0.96
	20+			0.94	
[1970, 1975)	[0, 5)			0.92	0.98
	[5, 10)		0.92	0.97	0.94
	[10, 15)		0.92	0.96	0.95
	20+			0.94	
[1975, 1980)	[0, 5)			0.93	0.94
	[5, 10)		0.95	0.93	0.96
	[10, 15)		0.96	0.94	0.97
	20+			0.95	

iterations, with a burn in of 100, were judged sufficient for accurate inference.

4.2. Results and Discussion

Point estimates of the coverage of the marginal 0.95 posterior probability intervals are shown in Table 4. These are all close to 0.95.

In practical applications with real data sets, where the true parameter values are unknown, interest will be in interval estimates for the demographic parameters. These should be based on the joint posterior distribution. For illustration, we have plotted central marginal credible intervals for a selection of parameters and transformations that might be of interest based on the MCMC sample from a single replicate of the simulation study. For comparison, we have also plotted the true parameter values used throughout the simulation and the noisy measurements generated under the model.

Marginal central credible intervals can be plotted for age-specific parameters as has been done for age-specific fertility rates in Figure 2a. Credible intervals for any function of the age-specific parameters can be obtained immediately by transforming each vector of age-specific values in the MCMC sample and computing the sample quantiles. We show TFR in Figure 2b. This is obtained from the age-specific rates by summing them and multiplying by the width of the sub-intervals of the reconstruction period, which is five in this case. In a stationary population (Preston et al. 2001) subject to the survival proportions, $s_{0,t}, \ldots, s_{A_+,t}, s_{A_{++},t}$,

$$\mathrm{LEB} = 5\sum_{a=0}^{A_+}\prod_{i=0}^{a} s_{i,t} + \left(\prod_{i=0}^{A_+} s_{i,t}\right)\big(s_{A_{++},t}/(1-s_{A_{++},t})\big).$$

We plot this in Figure 2c. The total net number of migrants (Figure 2d) can be calculated by solving (1) for $\boldsymbol{g}_t$ and summing over age.

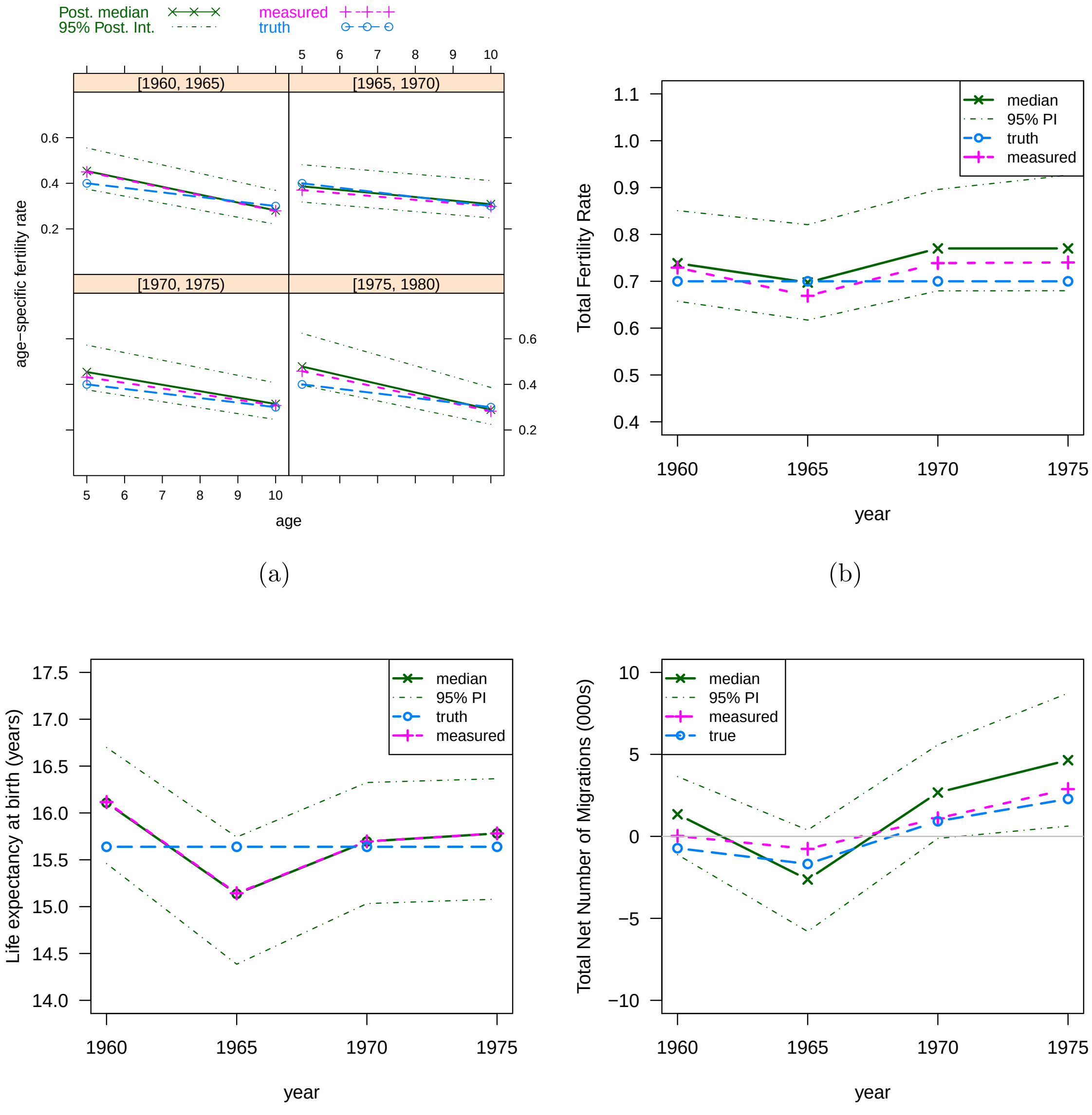
5. RECONSTRUCTION OF THE POPULATION OF BURKINA FASO, 1960–2000

We now illustrate the method by reconstructing the population of Burkina Faso from 1960 to 2000.

5.1. Data

5.1.1. Population Counts

Population count measurements in exact years 1960, 1975, 1985, 1995 and 2005 by sex were taken from United Nations (2009) which are based on a 1960–1961 demographic survey and censuses in 1975, 1985, 1996 and 2006. The United Nations (UN) figures are preferred over the raw census counts because important adjustments were made for underenumeration. This form of bias is more common in certain age-groups



(a) (b)

(c) (d)

Figure 2. Ninety-five Percent Central Credible Intervals for Selected Parameters From a Single Replication of the Simulation Study. (a) Age specific fertility rate. (b) Total fertility rate. (c) Life expectancy at birth in the stable equivalent population. (d) Total net migration.

and efforts to reduce it are based on post-censal surveys.

5.1.2. Fertility

Estimates of age-specific fertility rates for Burkina Faso based on data from recent births (in the preceding 12 months) and retrospective birth histories were obtained from various sources. The recent births data came from the 1960 and 1991 demographic surveys, as well as from the 1976 census post-enumeration survey and the 1985, 1996 and 2006 censuses themselves. The retrospective birth histories were taken from the 1992–93, 1998–99 and 2003 Demographic and Health Surveys (DHSs) and refer to the preceding 20 years. In addition, for each of these data sources, adjusted fertility rates for the most recent period were also estimated using women's lifetime parity information and the Brass-Feeney P/F ratio method (Brass 1964, Feeney 1996; see also United Nations 1983, Ch. II, Sect. B).

Alkema et al. (2008) studied estimates of TFR for seven West African countries, including Burkina Faso, and found noticeable disagreement among estimates from the various data sources. This suggested the need to account for potential bias and to estimate the variability of the estimates. Therefore, our prior median estimates of age-specific fertility were based on Alkema et al.'s (2008) median bias-adjusted estimates of TFRs for Burkina Faso for the five-year increments between, and inclusive of, the years 1962–1997. These were chosen as they are the mid-points of the five-year sub-intervals [1960, 1965), [1965, 1970), . . . , [1995, 2000) which span the reconstruction period. Since the latest estimate given by Alkema et al. (2008) is for 2001, this was used as the estimate for the period [2000, 2005). These were then disaggregated into age-specific rates by applying an age-specific fertility pattern. These patterns, which sum to one, indicate the share of fertility attributable to each age-group. Therefore, the final age-specific fertility rates are the product of TFR and the age pattern. The patterns were determined through a process we now describe.

Since we require average fertility rates for the sub-intervals [1960, 1965), ..., [2000, 2005), we grouped the available estimates by the sub-interval into which they fell. Data of this kind are often summarized by a single series of age-specific values per sub-interval using a relational model or a smoothing technique. Relational models take a fixed age-pattern, often derived from a combination of data collected in similar populations and theory about the underlying social and biological processes, and adjust it to fit the observed data. The method of Coale and Trussell (1974), updated by Xie (1990) and Xie and Pimentel (1992), is an example. However, the validity of these methods rests, in part, on an appropriate choice of model age pattern. Data-driven smoothing techniques avoid this problem, albeit at the cost of not modeling the underlying mechanisms. Since we are not primarily concerned with such mechanisms, we used *loess* (Cleveland 1979, Cleveland, Grosse, and Shyu 1992) to smooth the age-specific fertility patterns within five-year sub-interval. The *loess* method performs a series of locally weighted regressions and is implemented in the *R* function with the same name. Smoothing within five-year sub-interval (Figure 3) yielded trends that were also sensible, *a priori*, when viewed by five-year age group (Figure 4). No age-pattern data were available for the period [1965,1970). To generate a prior for this period, we assumed that the [1960,1965) held over this period but used Alkema et al.'s (2008) median TFR estimate for 1967.

5.1.3. Mortality

Abridged life tables for Burkina Faso can be computed from data on recent household deaths in years for which data is available. However, potential biases arise due to the omission of deaths, recall period errors, age heaping and age exaggeration by survey respondents. The approach favored by the UN in this context has been to use

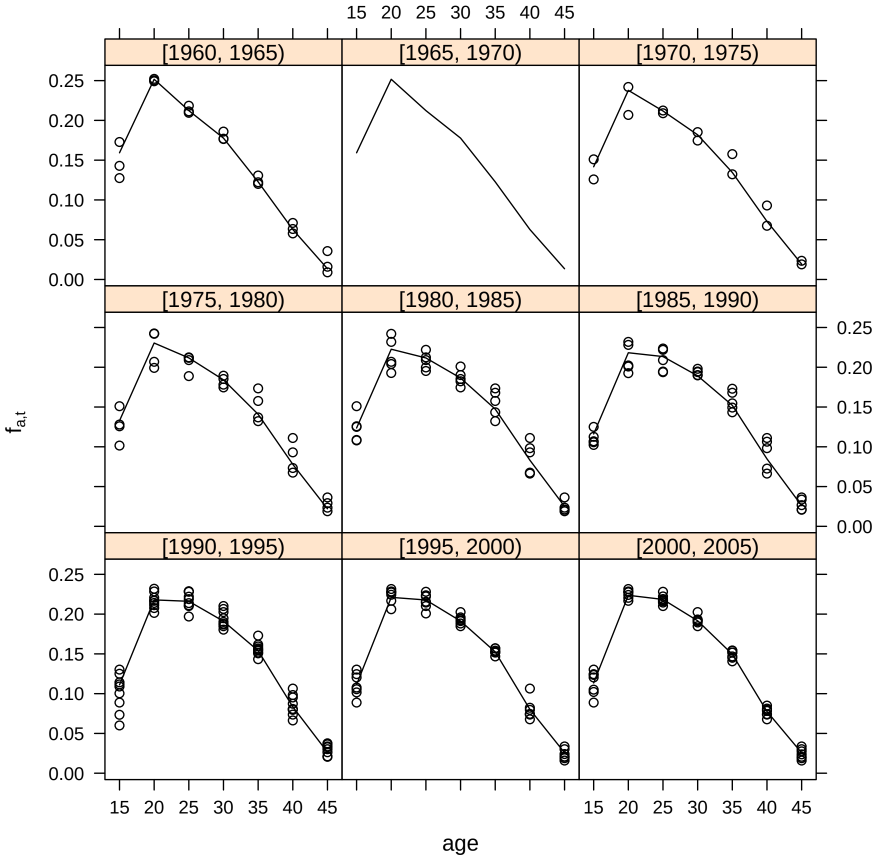


Figure 3. Data Points for the Age-Specific Fertility Patterns of Burkina Faso Women, 1960–2000, Grouped By on Five-year Sub-Interval. The lines are the within-time loess smooths.

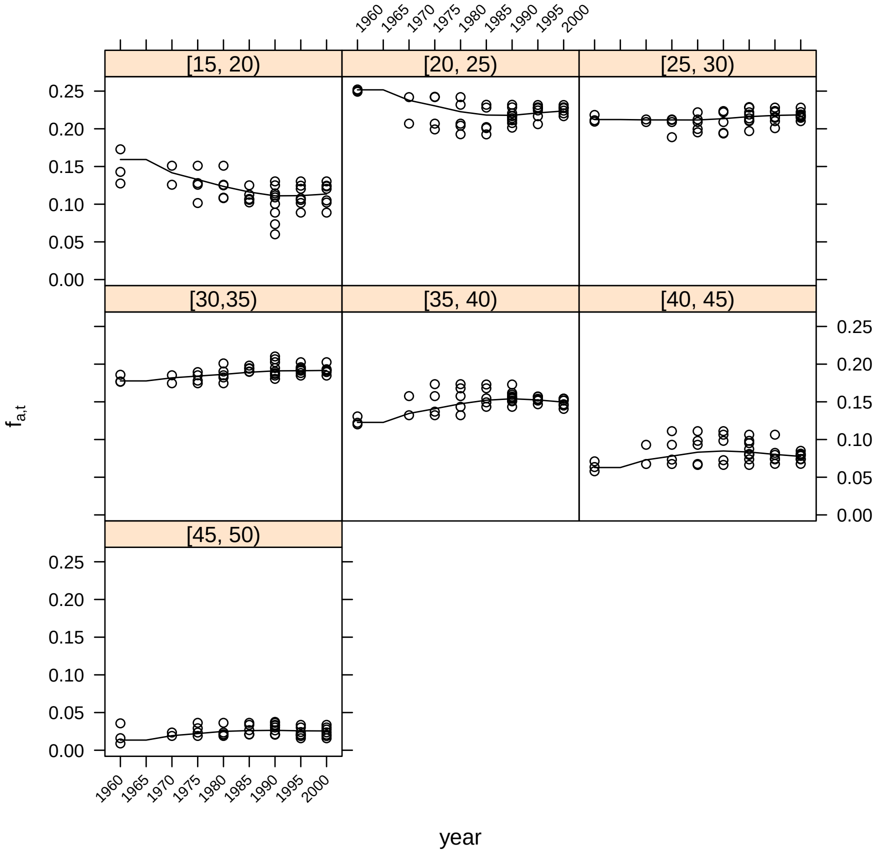


Figure 4. Data Points for the Age-Specific Fertility Patterns of Burkina Faso Women, 1960–2000, Grouped by Five-year Age Group. The lines are the within-time loess smooths, the same as those plotted in Figure 3.

a Brass two-parameter relational logit model (Brass 1971), with the Timæus Sahelian standard mortality pattern (Timæus 1999), to estimate a complete set of abridged life tables for each five-year sub-interval of the reconstruction period. Brass's method treats the logit of the life table for the population of interest as a linear function of the logit of an appropriately selected standard life table. The intercept and slope are estimated using ordinary least squares (OLS).

Brass's model was fitted to robust estimates of under-five mortality and adjusted estimates of adult mortality. Under-five mortality estimates were based on three types of data: (i) recent household deaths from the 1960–1961 and 1991 national demographic surveys and the 1976, 1985, 1996 and 2006 censuses; (ii) births and deaths to under-fives calculated from maternity-history data from the 1992–1993, 1998–1999 and 2003 DHSs; and (iii) data on children ever-born and surviving classified by age of mother (and the South model of the Coale-Demeny Model Life Tables) from the data sources in (ii) as well as from UNICEF's Multiple Indicator Cluster Survey—Round 3 conducted in 2006. Estimates of adult mortality were based on four sources: (i) recent household deaths data (unadjusted and adjusted for underregistration using the growth-balance and synthetic-extinct generation methods) from the 1960–1961 and 1991 national demographic surveys and the 1976, 1985, 1996 and 2006 censuses; (ii) parental orphanhood from the 1993 and 2003 DHSs and the 2006 census; (iii) sibling deaths from the 1998–1999 and 2003 DHSs; and (iv) intercensal survivorship from successive census age distributions (smoothed and not smoothed) for the periods 1976–1985, 1985–1996 and 1996–2006.

The usual approach to obtaining age-specific survival proportions from the resulting life-tables is to assume the relationships $s_{0,t} = L_{0,t}/(5 \cdot l_{0,t})$, $s_{a,t} = L_{a,t}/L_{a-5,t}$ for $a = 5, \ldots, 80$ and $s_{85+,t} = \sum_{a=85}^{\infty} L_{a,t} / \sum_{a=80}^{\infty} L_{a,t}$ which hold in the stationary equivalent population (Preston et al. 2001). $L_{a,t}$ is the number of person years lived between years t and $t+5$ by those with ages in $[a, a+5)$ in year t and $l_{0,t}$ is the radix of the life table or annual number of births in the stable equivalent population. While we need survival ratios for the CCMPP, demographers more often deal with the mortality rates, $m_{a,t} \equiv d_{a,t}/L_{a,t}$, where $d_{a,t}$ is the number of deaths to the age group $[a, a+5)$ in the life table operating during the sub-interval $[t, t+5)$. In the stationary equivalent population, $d_{a,t} = s_{a,t}n_{a,t} - n_{a,t}$ so that the $s_{a,t}$ can be easily re-expressed as mortality rates.

The stationary equivalent device is used frequently in demography. In stationary populations the birth rate and age-specific death rates are constant over time and the population is closed to migration. However, we emphasize that stationary population conditions are not assumed to hold in the estimation process. The device is invoked only to derive the measurements, $s^*_{a,t}$, and as a way to re-express the survival proportions in a form more familiar to demographers.

5.1.4. Migration

Estimates of migration for many countries, even those with well-resourced official statistics systems, are often unavailable, unreliable or available only at the whole population level. Nevertheless, some estimates of migrant flows for Burkina Faso by broad age-group and sex are given by Condé (1980) for the period 1960–1975. In addition, whole-population estimates for 1960–2005 are available from United Nations (2009) and United States Census Bureau (2009) which indicate sustained net out-migration over the period 1960–2000 followed by a brief reversal from 2000–2005. Condé (1980) implies that the proportion of migration due to females is low. Therefore, we made the *a priori* assumptions that migration was concentrated among those in the age range 15–50 and that there was net out-migration for the period 1960–2000 but net in-migration from 2000–2005. The mean magnitude for females was estimated at ±7 percent based on Condé's (1980) figures.

5.2. Results

The parameters α_v and β_v, $v \in \{n, f, s, g\}$, were set to the same values as in the simulation study (Table 3); the MAEs given there were based on expert opinions provided by UNPD analysts for the case of Burkina Faso. Start values were also chosen in the same way as for the simulation study. Chains of length 15,000 with a burn-in of 500 were found to be sufficient.

Central marginal 95% credible intervals and posterior medians for the population counts and age-specific fertility rates are shown in Figures 5 and 6 for illustration. Similar plots can be constructed for age specific mortality rates, migration rates and population counts in the years following t_0. In Bayesian terminology, the latter are the "prior predictive" distributions of $\boldsymbol{n}_t$.

Central marginal credible intervals for (a) TFR, (b) total net number of migrants, (c) probability of death before age 5 (${}_5q_0$) and (d) LEB are shown in Figure 7. The interval widths for TFR over the reconstruction period are consistently equivalent to about plus or minus half a child. There was a decline in TFR from 7.9–7.0 children to 6.8–6.0 children per woman over the period 1980–2005. There was a strong upward trend in life expectancy from 33.5–37.1 to 50.9–53.8 years is implied over the period 1960–2005. While the posterior medians for the total net number of migrants suggest a change from a net loss to a net gain between 1995 and 2000, all 0.95 credible intervals contain zero. Our interpretation of this result is that there is not enough information on migration to clearly indicate a trend.

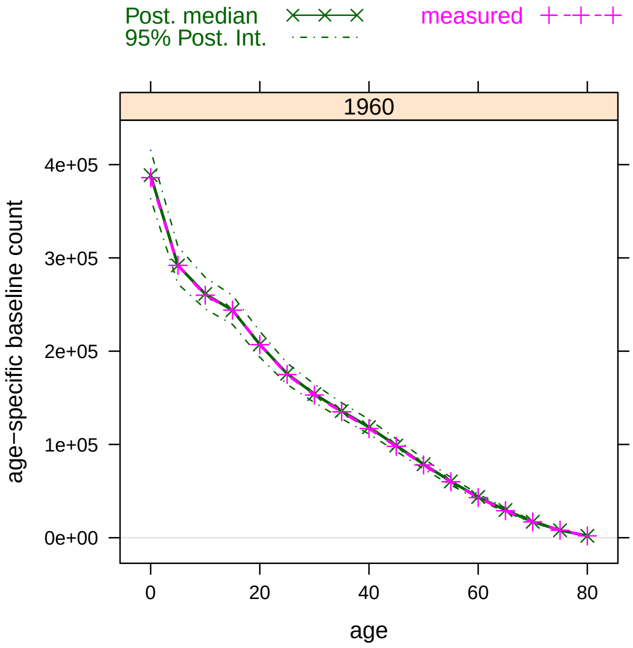


Figure 5. Ninety-five Percent Central Marginal Posterior Intervals for Initial Age-specific Population Counts for the Female popUlation of Burkina Faso, 1960. Also shown are the measured values.

6. DISCUSSION

We have described a method for reconstructing past populations by age and sex which is designed to work with the type of data commonly collected in modern demographic surveys and censuses. Population dynamics are modeled by the well-known CCMPP and measurement error is accounted for in a coherent, fully probabilistic manner through a Bayesian hierarchical model. Inference is based on the joint posterior distribution of all parameters.

Lee (1971, 1974) and Oeppen (1993) proposed deterministic methods of population reconstruction. We have assumed a deterministic model only for the population dynamics. That is, given the true vital and migration rates, the evolution of the population is modeled deterministically. In contrast, Bertino and Sonnino (2003) give a method wherein the population dynamics are stochastic and the vital rates function

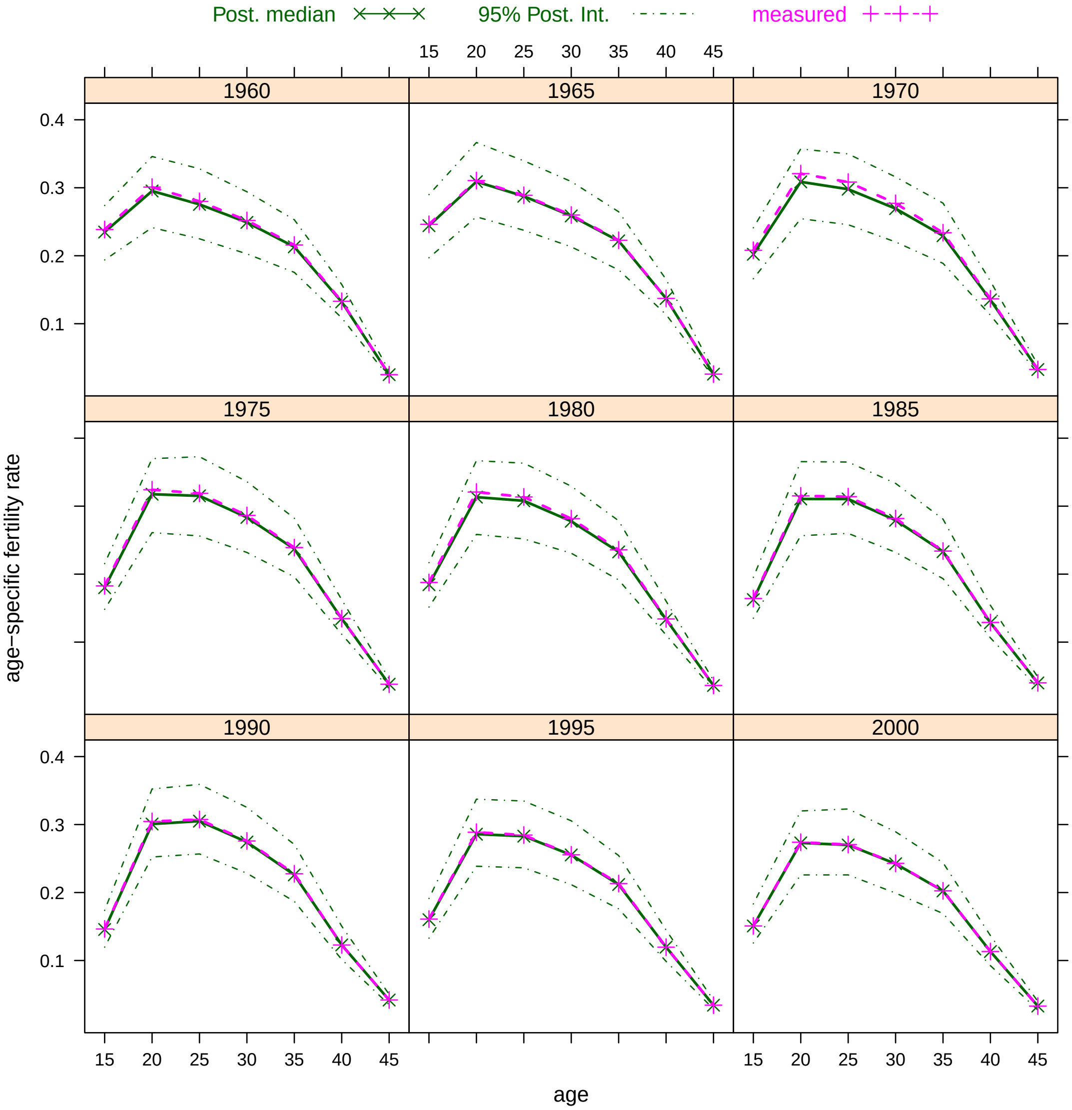
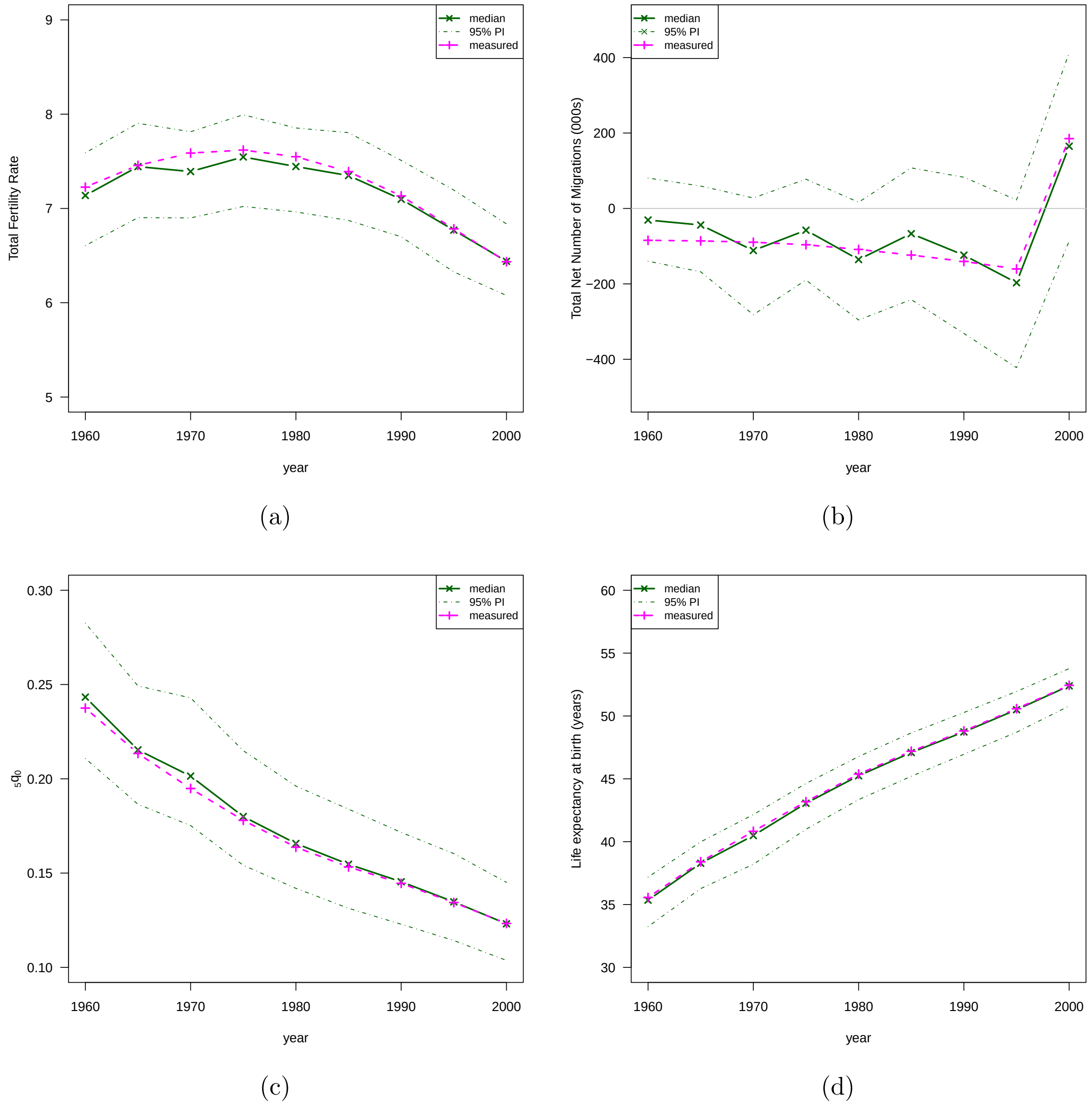


Figure 6. Ninety-five Percent Central Marginal Posterior Intervals for Age-specific Fertility Rates for the Female Population of Burkina Faso, 1960–2005. Also shown are the measured values.



(a) (b) (c) (d)

Figure 7. Ninety-five Percent Central Credible Intervals for Selected Age-summarized Parameters for the Female Population of Burkina Faso, 1960–2005. (a) Total Fertility Rate. (b) Total Net Migration. (c) Child Mortality ($_5q_0$) in the Stable Equivalent Population. (d) Life Expectancy at Birth in the Stable Equivalent Population.

as mean parameters. We agree with Cohen (2006) and expect that, in our applications at least, variation due to measurement error will overwhelm any additional variation arising from a stochastic population dynamics model.

Daponte, Kadane, and Wolfson (1997) use an approach similar to ours to construct a counterfactual history of the Iraqi Kurdish population between 1977 and 1990. They, too, represented errors in the measurement of demographic parameters as the standard deviations of probability distributions. However, they used a low dimensional parameterization of mortality and fixed the age patterns of fertility.

Girosi and King (2008) use Bayesian models with covariates to forecast mortality. Our aim was not to produce forecasts, nor model dependence on covariates. Instead, we have shown how information only about the parameters themselves can be used to produce probabilistic distributions summarizing uncertainty due to measurement error. Where useful covariate information is available, we suggest that this be applied in a preprocessing stage, for example to remove the bias from the age-specific estimates which our model takes as input and assumes are median-unbiased. Our method would then be used to ensure that final estimates of multiple parameters are coherent in that they satisfy the basic demographic accounting relationships.

In previous work, migration has been handled in different ways. Lee's inverse projection method estimates it as a residual if censuses are available at intermediate years in the reconstruction interval. Alho (1992) added it as extra error in measuring survival. Here, migration is treated explicitly and age-time specific estimates are available as for the other parameters.

Similar models have been used extensively in demographic studies of marine life and complex population dynamics models have been developed by fisheries researchers (e.g., Quinn and Deriso 1999). As is the case in human demography, multiple sources of data informing the same demographic parameters often exist. This led Raftery, Givens, and Zeh (1992, 1995), Givens, Raftery, and Zeh (1993) and McAllister et al. (1994) to develop Bayesian approaches to synthesizing these in a coherent manner.

A large body of work also exists on the dynamics of land animal populations. As with fisheries research, data commonly come from mark-recapture or mark-recovery studies, but radio-telemetry and age-at-harvest data are also common; Seber (1982) is a classic reference and more recent reviews are Pollock (1991) and Schwarz and Seber (1999). Bayesian approaches to the analysis of this type of data were suggested at least as early as Gaskell and George (1972) but Vounatsou and Smith (1995) were among the first to take advantage of modern computers and MCMC methods to simultaneously estimate several parameters. Subsequently, a large body of literature developed; Brooks, Catchpole, and Morgan (2000a), Brooks et al. (2002) provided reviews while Barry et al. (2003), Conn et al. (2008) and Corkrey et al. (2008) are just a few examples of more recent studies.

An issue given some consideration in this literature (and the demography literature; see Lee 1985, 1993) is the fact that the population dynamics models are typically over-parameterized. This causes problems for likelihood based inference because the likelihood surface then contains ridges (e.g., Catchpole, Kgosi, and Morgan 2001). Nevertheless, in one of the models they studied, Brooks et al. (2000b) show that a Bayesian approach can yield sensible estimates.

Naturally, there are ways in which our framework can be refined. An extension to two-sex populations is an obvious example. We expect this to be straightforward conceptually, although technical issues associated with estimation may arise. With two sexes, we might also attempt to estimate the SRB parameter.

We made the simplifying assumption of constant variance across age and time for each demographic parameter on the log scale. Allowing uncertainty estimates to vary would allow for the fact that more is typically known about infant and child mortality than old-age mortality, for example. However, the benefits of this extension would have to be weighed against the expense of having to estimate more parameters. This might be beneficial if estimation is done simultaneously for groups of countries. If the groups are chosen to be demographically similar, it might be realistic to set some of the variance parameters constant across countries but allow for variation over time or age.

Alternatively, a small number of variance parameters could be chosen representing different degrees of uncertainty, for example, $\sigma^2_{SML} < \sigma^2_{MED} < \sigma^2_{LGE} < \sigma^2_{XL}$. These would then be assigned to the age-specific parameters following the same reasoning we have advocated above. An advantage is that the number of parameters can be determined by the analyst as opposed being dependent on the number of age groups or sub-intervals.

REFERENCES

Alho, J. M. (1992), "The Magnitude of Error Due to Different Vital Processes in Population Forecasts," *International Journal of Forecasting*, 8, 301–314.

Alkema, L., Raftery, A. E., Gerland, P., Clark, S. J., and Pelletier, F. (2008), "Estimating the Total Fertility Rate from Multiple Imperfect Data Dources and Assessing its Uncertainty," Working Paper 89, University of Washington, Center for Statistics and the Social Sciences.

Barry, S. C., Brooks, S. P., Catchpole, E. A., and Morgan, B. J. T. (2003), "The Analysis of Ring-Recovery Data Using Random Effects," *Biometrics*, 59, 54–65.

Bertino, S. and Sonnino, E. (2003), "The Stochastic Inverse Projection and the Population of Velletri (1590-1870)," *Mathematical Population Studies*, 10, 41–73.

Booth, H. (2006), "Demographic Forecasting: 1980 to 2005 In Review," *International Journal of Forecasting*, 22, 547–581.

Boyle, P. P. and Grada, C. O. (1986), "Fertility Trends, Excess Mortality, and the Great Irish Famine," *Demography*, 23, 543–562.

Brass, W. (1964), "Uses of Census and Survey Data for the Estimation of Vital Rates," Paper prepared for the African Seminar on Vital Statistics, Addis Ababa.

— (1971), *Biological Aspects of Demography*, London: Taylor and Francis Ltd., chap. On the Scale of Mortality, pp. 69–110.

Brooks, S. P., Catchpole, E. A., and Morgan, B. J. T. (2000a), "Bayesian Animal Survival Estimation," *Statistical Science*, 15, 357–376.

Brooks, S. P., Catchpole, E. A., Morgan, B. J. T., and Barry, S. C. (2000b), "On the Bayesian Analysis of Ring-Recovery Data," *Biometrics*, 56, 951–956.

Brooks, S. P., Catchpole, E. A., Morgan, B. J. T., and Harris, M. P. (2002), "Bayesian Methods for Analysing Ringing Data," *Journal of Applied Statistics*, 29, 187–206.

Catchpole, E. A., Kgosi, P. M., and Morgan, B. J. T. (2001), "On the Near-Singularity of Models for Animal Recovery Data," *Biometrics*, 57, 720–726.

Cleveland, W. S. (1979), "Robust Locally Weighted Regression and Smoothing Scatterplots," *Journal of the American Statistical Association*, 74, 829–836.

Cleveland, W. S., Grosse, E., and Shyu, W. M. (1992), "Local Regression Models," in *Statistical Models in S*, eds. Chambers, J. and Hastie, T., Pacific Grove, Calif: Wadsworth & Brooks/Cole, chap. 8.

Coale, A. J. and Trussell, J. (1974), "Model Fertility Schedules: Variations in the Age Structure of Childbearing in Human Populations," *Population Index*, 40, 185–258.

Cohen, J. E. (2006), "Stochastic Demography," in *Encyclopedia of Statistical Sciences*, John Wiley and Sons, Inc., article Online Posting Date: August 15, 2006.

Condé, J. (1980), "Migration in Upper Volta," in *Demographic Aspects of Migration in West Africa*, eds. Zachariah, K. C. and Condé, J., Washington D. C.: World Bank, vol. 2 of *Staff Working Paper, No. 415*.

Conn, P. B., Diefenbach, D. R., Laake, J. L., Ternent, M. A., and White, G. C. (2008), "Bayesian Analysis of Wildlife Age-at-Harvest Data," *Biometrics*, 64, 1170–1177.

Corkrey, R., Brooks, S., Lusseau, D., Parsons, K., Durban, J. W., Hammond, P. S., and Thompson, P. M. (2008), "A Bayesian Capture-Recapture Population Model With Simultaneous Estimation of Heterogeneity," *Journal of the American Statistical Association*, 103, 948–960.

Daponte, B. O., Kadane, J. B., and Wolfson, L. J. (1997), "Bayesian Demography: Projecting the Iraqi Kurdish Population, 1977-1990," *Journal of the American Statistical Association*, 92, 1256–1267.

Feeney, G. (1996), "A New Interpretation of Brass' P/F Ratio Method Applicable When Fertility is Declining," `http://www.heart-intl.net/HEART/HIV/Comp/InterpretationofFertiline.pdf`.

Gaskell, T. J. and George, B. J. (1972), "A Bayesian Modification of the Lincoln Index," *Journal of Applied Ecology*, 9, 377–384.

Geman, S. and Geman, D. (1984), "Stochastic Relaxation, Gibbs Distributions, and the Bayesian Restoration of Images," *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, PAMI-6, 721–741.

Girosi, F. and King, G. (2008), *Demographic Forecasting*, Princeton, NJ: Princeton University Press.

Givens, G. H., Raftery, A. E., and Zeh, J. E. (1993), "Benefits of a Bayesian Approach for Synthesizing Multiple Sources of Evidence and Uncertainty Linked by a Deterministic Model," *Report of the International Whaling Commission*, 43, 495–500.

Goodkind, D. and West, L. (2001), "The North Korean Famine and its Demographic Impact," *Population and Development Review*, 27, 219–238.

Hastings, W. K. (1970), "Monte Carlo Sampling Methods Using Markov Chains and Their Applications," *Biometrika*, 57, 97–109.

Heuveline, P. (1998), "'Between One and Three Million': Towards the Demographic Reconstruction of a Decade of Cambodian History (1970–79)." *Population Studies*, 52, 49–65.

Hill, K., Pande, R., Mahy, M., and Jones, G. (1999), *Trends in Child Mortality in the Developing World: 1960 to 1996*, New York, NY: UNICEF.

Keilman, N. (1998), "How Accurate Are the United Nations World Population Projections?" *Population and Development Review*, 24, 15–41.

Lee, R. D. (1971), "Econometric Studies of Topics in Demographic History," Disseration in economics, Harvard University, Cambridge, Massachucetts.

— (1974), "Estimating Series of Vital Rates and Age Structures from Baptisms and Burials: A New Technique, with Applications to Pre-Industrial England," *Population Studies*, 28, 495–512.

— (1985), "Inverse Projection and Back Projection: A Critical Appraisal, and Comparative Results for England, 1539 to 1871," *Population Studies*, 39, 233–248.

— (1993), "Inverse Projection and Demographic Fluctuations," in *Old and New Methods in Historical Demography*, eds. Reher, D. S. and Schofield, R., Oxford, England: Clarendon Press, chap. 1, pp. 7–28.

— (1998), "Probabilistic Approaches to Population Forecasting," *Population and Development Review*, 24, 156–190.

Leslie, P. H. (1945), "On the Use of Matrices in Certain Population Mathematics," *Biometrika*, 33, 183–212.

— (1948), "Some Further Notes on the Use of Matrices in Population Mathematics," *Biometrika*, 35, 213–245.

Lewis, E. G. (1942), "On the Generation and Growth of a Population," *Sankhyā: The Indian Journal of Statistics (1933-1960)*, 6, 93–96.

McAllister, M. K., Pikitch, E. K., Punt, A. E., and Hilborn, R. (1994), "A Bayesian Approach to Stock Assessment and Harvest Decisions Using the Sampling/Importance Resampling Algorithm," *Canadian Journal of Fisheries and Aquatic Sciences*, 51, 2673–2687.

McCaa, R. and Barbi, E. (2004), "Inverse Projection: Fine-Tuning and Expanding the Method," in *Inverse Projection Techniques: Old and New Approaches*, eds. Barbi, E., Bertino, S., and Sonnino, E., Berlin: Springer-Verlag, pp. 11–28.

Merli, M. G. (1998), "Mortality in Vietnam, 1979-1989," *Demography*, 35, 345–360.

Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H., and Teller, E. (1953), "Equation of State Calculations by Fast Computing Machines," *Journal of Chemical Physics*, 21, 1087–1092.

Oeppen, J. (1993), "Back projection and Inverse Projection: Members of a Wider Class of Constrained Projection Models," *Population Studies*, 47, 245–267.

O'Hagan, A., Buck, C. E., Deaneshkhah, A., Eiser, J. R., Garthwaite, P. H., Jenkinson, D. J., Oakley, J. E., and Rakow, T. (2006), *Uncertain judgements: Eliciting Experts' Probabilities*, Statistics in Practice, London: Wiley.

Plummer, M., Best, N., Cowles, K., and Vines, K. (2006), "CODA: Convergence Diagnosis and Output Analysis for MCMC," *R News*, 6, 7–11.

— (2010), *CODA: Output Analysis and Diagnostics for MCMC.*, Comprehensive R Archive Network (CRAN), `www.r-project.org`.

Pollock, K. H. (1991), "Modeling Capture, Recapture, and Removal Statistics for Estimation of Demographic Parameters for Fish and Wildlife Populations: Past, Present, and Future," *Journal of the American Statistical Association*, 86, 225–238.

Preston, S. H., Heuveline, P., and Guillot, M. (2001), *Demography: Measuring and Modeling Population Processes*, Blackwell.

Quinn, T. J. and Deriso, R. B. (1999), *Quantitative Fish Dynamics*, New York, NY: Oxford University Press.

R Development Core Team (2010), *R: A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria, ISBN 3-900051-07-0.

Raftery, A. E., Givens, G. H., and Zeh, J. E. (1992), "Inference from a Deterministic Population Dynamics Model About Bowhead Whale, *Balaena Mysticetus*, Replacement Yield," Technical Report 232, Department of Statistics, University of Washington, Seattle, WA.

— (1995), "Inference from a Deterministic Population Dynamics Model for Bowhead Whales," *Journal of the American Statistical Association*, 90, 402–416.

Raftery, A. E. and Lewis, S. M. (1996), "Implementing MCMC," in *Markov Chain Monte Carlo in Practice*, eds. Gilks, W. R., Richardson, S., and Spiegelhalter, D. J., London: Chapman & Hall, pp. 115–130.

Rosina, A. (2004), "Using Information on the Age Distribution of Deaths in Population Reconstruction: An Extension of Inverse Projection with Applications," in *Inverse Projection Techniques: Old and New Approaches*, eds. Barbi, E., Bertino, S., and Sonnino, E., Berlin: Springer-Verlag, pp. 29–38.

Schwarz, C. J. and Seber, G. A. F. (1999), "Estimating Animal Abundance: Review III," *Statistical Science*, 14, 427–456.

Seber, G. A. F. (1982), *The Estimation of Animal Abundance and Related Parameters*, London: Charles Griffin & Company Ltd., 2nd ed.

Timæus, I. M. (1999), "Notes on a Series of Life Table Estimates of Mortality in the Countries of the sub-Saharan Africa Region," Unpublished manuscript prepared for the World Health Organization.

UNICEF, WHO, The World Bank, and UN Population Division (2007), "Levels and Trends of Child Mortality in 2006: Estimates developed by the Inter-agency Group for Child Mortality Estimation," Working paper, New York, NY.

United Nations (1983), *Manual X: Indirect Techniques for Demographic Estimation*, no. 81 in Population Studies (Department of International, Economic and Social Affairs), New York, NY: United Nations, Sales No. E.83.XIII.2.

— (2009), "World Population Prospects: The 2008 Revision," `http://esa.un.org/unpd/wpp2008/index.htm`.

United States Census Bureau (2009), "International Data Base: Table 008, Crude Birth and Death, Net Migration, and Growth Rates, Births, Deaths, and Migrants, Burkina Faso, 1976–2050," `http://www.census.gov/ipc/www/idb/tables.html`.

Vounatsou, P. and Smith, A. F. M. (1995), "Bayesian Analysis of Ring-Recovery Data Via Markov Chain Monte Carlo Simulation," *Biometrics*, 51, 687–708.

Wrigley, E. A. and Schofield, R. S. (1981), *The Population History of England, 1541–1871: A Reconstruction*, Studies in Social and Demographic History, London: Edward Arnold.

Xie, Y. (1990), "What is Natural Fertility? The Remodelling of a Concept," *Population Index*, 56, 656–663.

Xie, Y. and Pimentel, E. E. (1992), "Age Patterns of Marital Fertility: Revising the Coale-Trussell Method," *Journal of the American Statistical Association*, 87, 977–984.